May 3d/47

AN

ELEMENTARY TREATISE

ON

ALGEBRA,

FOR THE USE OF STUDENTS

IN

HIGH SCHOOLS AND COLLEGES.

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SECOND EDITION.

BOSTON:

BENJAMIN B. MUSSEY.

1844.

PREFACE.

THE author of this treatise has endeavored to prepare a work which should sufficiently exercise the ability of most learners, without becoming, at the same time, repulsive to them by being excessively abstract. Some writers err in expecting too much, and others err, in an equal degree, by requiring too little of the student. What success has attended an attempt to attain a proper medium it is left for competent teachers to decide.

This work commences in the inductive manner, because that mode is most attractive to beginners. As the learner advances, and acquires strength to grapple with it, he meets with the more rigorous kind of demonstration. This course seems the most natural and effective. Induction is excellent in its place; but when an attempt is made to carry it into all the departments of an exact science, the result often shows, that the main object of study was misapprehended. The young frequently fail to deduce clearly the general principle from the particular instances which have engaged their attention.

Several parts of algebra, which are either omitted or not explained with sufficient distinctness in other works, have received particular attention in this. These parts treat of principles and operations, with which students rarely become familiar, but which are essential to a clear comprehension of the subject. Among these operations may be mentioned the separation of quantities into factors, finding the divisors of quantities, and the substitution of numbers in algebraic formulæ.

Most of the problems are original; others have been selected, which seemed the most appropriate.

Although this treatise is designed for students in the higher grade of seminaries, it is not beyond the reach of any, who have a good knowledge of arithmetic, and who are under the guidance of competent instructors. Should Articles 46, 58, 59, 153 and 154 be found too difficult for the beginner, on his first perusal of the book, they may be postponed for investigation in a review.

The writer is unwilling to close his remarks, without expressing his obligations to others, who have done so much to introduce into our country a natural and rational mode of studying mathematics. Among these none merits greater praise than Colburn; and his works have served as a guide in the composition of several others on the inductive plan. Day, Smyth, Davies and Peirce deserve also to be mentioned with great respect.

THOMAS SHERWIN.

English High School,
Boston, Sept. 10, 1841.

In this new edition of his work, the author would remark, that the few errors of the first edition have been carefully corrected; that a Key to the Algebra has been published; and that, in both the Algebra and Key, a marked distinction has been made between the full point when used as the sign of multiplication and when used as a decimal point; in the latter case, the type being inverted, and the sign consequently elevated.

T. S.

April 4, 1843.

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ELEMENTS OF ALGEBRA.

PRELIMINARY REMARKS.

Art. **1.** ARITHMETIC treats of numbers which have known and definite values; but Algebra makes use of symbols, which may represent known, unknown, or indeterminate quantities. These symbols are the letters of the alphabet.

Moreover, in Arithmetic, after an answer to a question has been obtained, it contains nothing in itself to show by what operations it was found. For instance, suppose the number 6 is ascertained to be the answer to a particular question; this exhibits no marks to show whether it was obtained by addition, multiplication, division, or by some other process or combination of processes; but the results of *pure Algebra*, that is, when both known and unknown quantities are represented by letters, always indicate, or may be made to indicate, the means by which they were produced.

Algebra enables us also to carry on a course of reasoning with much greater ease and rapidity than Arithmetic, and to arrive at the solution of problems, which, by the aid of Arithmetic alone, would be exceedingly difficult, if not impossible.

Art. **2.** We proceed to notice some of the signs, which most frequently occur in Algebra.

The sign $+$ is used to express *addition*, and is called *plus*, which signifies *more;* thus, $6+3$ is read 6 plus 3, and means

that 6 and 3 are to be added together, or indicates the *sum* of 6 and 3.

The sign — expresses *subtraction*, and is called *minus*, which signifies *less;* thus $8-3$ is read 8 minus 3, and means that 3 is to be subtracted from 8, or indicates the *difference* between 8 and 3.

Moreover, quantities having before them the sign $+$, expressed or understood, are called *positive;* those having before them the sign —, are called *negative quantities.*

Multiplication is represented by the sign $\times$; thus, 6×4 means that 6 and 4 are to be multiplied together, or indicates the *product* of 6 and 4. Sometimes a point between the quantities, as $6\,.\,4$, has the same signification.

Division is represented by the sign $\div$, or : ; but more frequently it is expressed in the form of a fraction; thus, $6\div 3$, $6:3$ and $\frac{6}{3}$, each signifies the division of 6 by 3, or indicates the *quotient* of that division.

To express *equality* we use two horizontal parallel lines, thus $=$; this is read *equal to, equals,* or by some words of similar import; for example, $6+4=10$ means that the sum of 6 and 4 is equal to 10, and is read 6 plus 4 equals 10.

Accordingly, $5\times 4+7=\frac{60}{2}-3$ means, that if 5 be multiplied by 4, and 7 be added to the product, the result will be the same as if 60 be divided by 2, and 3 be subtracted from the quotient.

The sign $>$, or $<$, is used to express the *inequality* of quantities; thus, $8>5$, or $5<8$, signifies that 8 is greater than 5, or that 5 is less than 8, the open end always being placed towards the greater quantity.

To represent unknown quantities, we use some of the last letters of the alphabet, as x, y, &c.; and to represent known quantities, we use some of the first letters, as a, b, c, &c.; although, in many problems of this work, known quantities are expressed by figures.

Art. **3**. There are some propositions, the truth of which is manifest, as soon as they are presented to the mind. These propositions are called *axioms;* the following are of this kind.

AXIOMS.

1. If the same quantity or equal quantities be *added* to equal quantities, the *sums* will be equal.

2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* by the same quantity or by equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same quantity or by equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be changed.

6. If a quantity be both *multiplied* and *divided* by another, its value will not be changed.

7. Two quantities, each of which is equal to a third, are equal to each other.

8. The whole of a quantity is greater than a part of it

9. The whole of a quantity is equal to the sum of all its parts.

SECTION I.

EQUATIONS OF THE FIRST DEGREE, HAVING ONLY UNKNOWN TERMS IN ONE MEMBER AND KNOWN QUANTITIES IN THE OTHER.

Art. 4. 1. An apple and an orange together cost 6 cents; but the orange cost twice as much as the apple. What was the price of each?

In this question, if we knew the price of the apple, we should, by doubling it, obtain that of the orange. The price of the apple, then, may be considered as the unknown quantity.

Suppose that x represents the number of cents given for the apple; twice as much, or the price of the orange, would be represented by $2x$.

Hence, $x + 2x = 6$. Putting the x's together, we have
$3x = 6$; one x will be $\frac{1}{3}$ as much;
therefore, $x = 2$ cents $=$ the price of the apple;
and $2x = 4$ cents $=$ the price of the orange.

Remark. Questions in Algebra may be proved as well as those in Arithmetic. The proof of the foregoing, would consist in adding the price of the apple to that of the orange, and ascertaining that their sum is 6 cents. Let the learner prove the correctness of his answers, as he advances.

A representation of the equality of quantities, is called an *equation*. Thus, $x + 2x = 6$ is an equation.

A *member* or *side* of an equation, signifies the quantity or quantities on the same side of the sign $=$, the *first member* being on the left, and the *second member* on the right hand side of this sign.

An equation of the *first degree* is one, in which the unknown quantities are neither multiplied by themselves nor by each other.

The separate parts of an algebraic expression affected by the signs $+$ and $-$, are called *terms*. Those terms which have no sign prefixed to them, are supposed to have the sign $+$, and a quantity is said to be *affected* by a sign, when it is immediately preceded by that sign, either expressed or understood. When the first term of a member of an equation, or of any algebraic quantity, is affected by the sign $+$, it is usual to omit writing the sign before that term; but the sign $-$ must always be written before any term affected by it. The equation, $x + 2x = 6$, consists of three terms, two in the first member and one in the second, and each of these terms is affected by the sign $+$.

The number written immediately before a letter, showing how many times the letter is taken, is called the *coëfficient* of that letter; thus, in the expressions, $3x$, $5x$, $7x$, the coëfficients of x are 3, 5 and 7. A letter which has no number written before it, is supposed to have 1 for its coëfficient; thus, x is the same as $1x$. Letters, as we shall see hereafter, may be used as coëfficients.

The process by which an equation is formed from the conditions of a question, is called *putting the question into an equation;* and the process by which the value of the unknown quantity is found from the equation, is called *solving the equation.*

2. Said A to B, my horse and saddle are worth \$110; but my

horse is worth 10 times as much as my saddle. Required the worth of each.

3. A man bought some corn and rye for 60 shillings, the corn at 4s. per bushel and the rye at 6s., and there was the same number of bushels of each. How many bushels were there of each?

Let x represent the number of bushels of each; then x bushels of corn at 4s. per bushel, will come to $4x$ shillings, and x bushels of rye at 6s. per bushel, will come to $6x$ shillings. Hence, $4x + 6x = 60$.

4. A man sold an equal number of oxen, cows and sheep; the oxen at $40 apiece, the cows at $15, and the sheep at $5; the whole came to $660. How many were there of each?

5. A woman bought some peaches, pears and melons for $1·10; the peaches at 1 cent apiece, the pears at 2, and the melons at 12; there were twice as many pears as melons, and three times as many peaches as pears. How many were there of each?

Let x represent the number of melons; then $2x$ will represent the number of pears, and $6x$, the number of peaches. At 1 cent each, $6x$ peaches come to $6x$ cents, $2x$ pears at 2 cents each will come to $4x$ cents, and x melons at 12 cents each will come to $12x$ cents; hence, $6x + 4x + 12x = 110$.

6. A gentleman hired a man and a boy to work a certain number of days, the man at 8s. and the boy at 4s. per day, and paid them $30. How many days were they employed, and how much did each receive?

7. Three numbers are in the proportion of 1, 2 and 3, and the sum of them is 630. What are these numbers?

The proportion of 1, 2 and 3, means that the second is twice, and the third three times as much as the first.

8. Divide 100 into three parts, in the proportion of 5, 7 and 8.

The proportion of 5, 7 and 8, means that the 2d is $\frac{7}{5}$, and the 3d $\frac{8}{5}$ as much as the 1st.

Suppose the 1st part $= 5x$, then the 2d will be $7x$, and the 3d, $8x$.

9. Two persons set out at the same time from two towns 150 miles apart, and travel towards each other till they meet, one at 8 miles an hour, and the other at 7. How many hours will they be on the road, and how far will each travel?

10. Three robbers, having stolen 48 guineas, quarrelled about the division of them, and each took as much as he could get; the first obtained a certain sum, the second twice as much, and the third as much as both the others. How many guineas did each obtain?

11. A gentleman wished to divide an estate of \$81000 between his wife and two sons, so that his wife should have \$4, as often as the elder son had \$3, and the younger \$2. How much would each receive?

12. A fortress has a garrison of 1200 men, a certain portion of whom are cavalry, three times as many artillerymen, and six times as many infantry. How many are there of each corps?

13. In fencing a field, three men, A, B and C, were employed. A could fence 9 rods a day, B 7, and C 5; B wrought twice as many days as A, and C five times as many as B. The distance round the field was 584 rods. How many days did each work, and how many rods of fence did each build?

14. A man bought three pieces of cloth for \$280. The second piece was twice as long as the first, and the third was as long as the first two. He gave \$4 a yard for the first piece, \$5 a yard for the second, and \$7 a yard for the third. Required the number of yards in each piece.

15. Four cows, 3 calves and 10 sheep cost \$112. A cow cost 5 times as much as a calf, and a calf cost twice as much as a sheep. Required the price of each.

16. A cistern holding 140 gallons, was filled with water by means of two buckets, the greater of which held twice as much as the less. The greater was emptied 7 times and the less 6 times. How many gallons did each bucket hold?

17. A boy being sent to market, bought some beef at 14 cents a pound, and twice as much mutton at 9 cents a pound. He was

intrusted with \$4 and brought back 80 cents. How many pounds of each kind of meat did he buy?

18. A man wished to pay \$60, with dollars, halves, quarters, and eighths, of each an equal number. How many coins of each kind would he require?

19. A man paid £144 in guineas at 21s. and crowns at 5s. each. There were three times as many crowns as guineas. Required the number of each.

20. A man on a journey traveled twice as far the 2d day as he did the 1st; on the 3d day, as far as he did the first two days; on the 4th day, as far as he did the first three days; and on the 5th day, half as far as on the 4th. The whole distance traveled was 150 miles. How far did he go each day?

21. A merchant exchanged rye at 7s. and wheat at 9s. a bushel, of each the same quantity, for 32 bushels of corn at 4s. a bushel. How many bushels of rye and wheat were given in exchange?

22. A drover bartered 6 oxen and 10 cows for a farm of 50 acres at \$11 per acre. He reckoned each ox worth as much as two cows. What price was assigned to an ox and a cow respectively?

Art. **5**. In the preceding questions, x's, that is, unknown quantities, have been found only in the first member of the equation, and they have all been affected by the sign $+$; and we perceive, that, after an equation was formed, the first step was to reduce or combine all the unknown quantities into one term, which is done by adding the coëfficients; after which, the value of the unknown quantity was found by dividing both members by the coëfficient of the unknown quantity.

SECTION II.

EQUATIONS OF THE FIRST DEGREE, HAVING UNKNOWN TERMS IN ONE MEMBER ONLY, AND KNOWN TERMS IN BOTH MEMBERS.

Art. **6**. Two brothers had together \$20, but the elder had two dollars more than the younger. How much money had each?

Let x represent the number of dollars the younger had;
then $x+2=$ the number of dollars the elder had;
consequently, $x+x+2=20$; or, combining the x's,

$$2x+2=20.$$

Now as the two members are equal, we can subtract 2 from each, and the remainders will be equal (ax. 2, Art. **3**); 2 subtracted from $2x+2$, leaves $2x$, and 2 subtracted from 20, leaves 18; hence, $2x=18$;

$$x=9, \text{ number of dollars the younger had,}$$
$$\text{and } x+2=11, \text{ number of dollars the elder had.}$$

Instead of actually subtracting 2 from the second member at once, we may subtract it from the first, and represent it as subtracted from the second; thus, $2x=20-2$; now performing the subtraction indicated, we have $2x=18$, the same as before. The equation $2x=20-2$ is obtained from $2x+2=20$ merely by removing the 2 to the other side of the sign $=$, and changing its sign from $+$ to $-$.

Art. **7**. Removing a term from one member of an equation to the other, is called *transposing* that term, or *transposition*. *Any term, therefore, affected by the sign $+$, may be transposed, if this sign be changed to $-$.*

1. Two men, A and B, hired a house for \$650, of which A paid \$150 more than B. What did each pay?

Let x represent the number of dollars B paid.

Then $x+150$ will represent the number A paid.

Hence, $x+x+150=650$. Reducing,

$$2x+150=650; \text{ transposing 150,}$$
$$2x=650-150; \text{ reducing the 2d member,}$$
$$2x=500,$$
$$x=\$250=\text{what B paid.}$$
$$x+150=\$400=\text{what A paid.}$$

2. Two men possess together \$56000, but the second has \$10000 more than the first. How much money has each?

3. Two towns are at unequal distances and in opposite directions from Boston; the distance between these towns is 230

miles, but one is 10 miles more than twice as far from Boston as the other. What is the distance of each from that city?

4. The sum of the ages of A, B and C is 100 years; but B's age is twice that of A and 5 years more, and C's age is equal to the sum of A's and B's. Required the age of each.

5. A grocer wishes to make a mixture of four kinds of tea, so that there shall be 6 lbs. more than twice as much of the 2d kind as of the 1st, as many lbs. of the 3d as there are of the first two, and as many of the 4th as there are of all the others; the whole mixture is to contain 120 lbs. How many lbs. must there be of each sort?

6. A man has five sons, each of whom is two years older than his next younger brother, and the amount of their ages is 50 years. What is the age of each?

7. A merchant bought 10 pieces of cloth for $331; 5 pieces were blue, 3 green, and 2 black; a piece of green cost $2 more than one of black, and a piece of blue $3 more than one of green. How much did each kind cost per piece?

8. Says A to B, my age is 10 years more than yours, and twice my age added to three times yours, makes 120 years. Required the age of each.

9. A gentleman leaves an estate of $10000, to be divided between his three daughters and two sons, in the following manner, viz: the daughters are all to share equally, but the elder son is to have $1000 more than twice as much as the younger, and the younger exactly twice as much as one of the daughters. What is the share of each?

10. A laborer undertook to reap 6 acres of wheat and 10 acres of oats for $\$21\frac{2}{3}$, or 130 shillings; but he was to have 3s. more an acre for the wheat than for the oats. What was the price of reaping an acre of each?

Let x shillings represent the price of reaping the wheat per acre.
Then $x-3$ will be the price of reaping the oats per acre.
Six acres of wheat will cost $6x$ shillings;
and ten acres of oats will cost $10x-30$ shillings.

Hence, $6x + 10x - 30 = 130$. Reducing,

$$16x - 30 = 130.$$

By adding 30 to each member (ax. 1, Art. **3**), the equation becomes

$16x - 30 + 30 = 130 + 30$, or,

$16x = 130 + 30$, since $16x - 30 + 30$ is the same as $16x$ (ax. 5); hence, $16x = 160$,

therefore, $x = 10$,

and $x - 3 = 7$. Ans. wheat 10s., oats 7s. per acre.

Most of the preceding questions in this section, may be solved in a similar way.

The equation $16x = 130 + 30$ is obtained from $16x - 30 = 130$, merely by removing the 30 to the second member of the equation, and changing its sign from $-$ to $+$.

Art. **8**. *Hence, any term affected by the sign $-$, may be transposed from one member to the other, if its sign be changed to $+$*; for, this is adding the same quantity to each member (ax. 1). This principle, together with that established in Art. **7**, gives the following general

RULE FOR TRANSPOSITION.

Art. **9**. *Any term may be transposed from one member of an equation to the other, care being taken to change its sign from $-$ to $+$, or from $+$ to $-$.*

It may be remarked, that the value of every such expression as, $1 - 1$, $2 - 2$, $3 - 3$, &c., or $x - x$, $4x - 4x$, $a - a$, $5a - 5a$, &c., is 0 or nothing; that is, the plus and minus quantities equal in value cancel each other.

Moreover, when quantities are connected by the signs $+$ and $-$, it is of no importance in what order they stand, provided they have their proper signs prefixed to them; thus, $3 + 7 - 2$ may be written $7 + 3 - 2$, or $-2 + 7 + 3$, the value of each expression being 8.

When the first term is affected by the sign $+$, it is usual to omit writing that sign; but the sign $-$ must never be omitted. The learner cannot be too careful with regard to the signs, as a mistake in the sign occasions an error equal to twice the

value of the term affected by it; thus, $12+3$ is equal to 15, and $12-3$ is equal to 9; now the difference between 15 and 9 is 6 or twice 3.

We perceive also, that when a quantity consisting of several terms, as $x-3$, is to be multiplied, each term must be multiplied and the same signs retained; thus 10 times $x-3$ is $10x-30$; in like manner, 7 times $12-3x$ is $84-21x$.

1. At a certain election, two persons were voted for; but the candidate chosen had a majority of 87, and the whole number of votes was 899. How many votes had each?

2. In a manufactory, 205 persons, men, boys and girls, are employed; there are four times as many boys as men, and 20 less than ten times as many girls as boys. How many of each are employed?

3. A general, on reviewing his troops, found he had in all 2300 men, of whom a certain portion were cavalry, three times as many riflemen, and 100 less than four times as many infantry as riflemen. How many were there of each?

4. Four men, A, B, C and D, enter into partnership. A contributes a certain sum, B three times as much, C twice as much as A and B both, and D as much as the other three wanting \$1000. The whole sum invested was \$65000; how much did each put in trade?

5. Divide \$491 among three persons, A, B and C, so that A shall have \$270 more, and B \$100 less than C.

Suppose $x=$ C's share. Then $x+270=$ A's share, and $x-100=$ B's share. Hence, $x+x+270+x-100=491$.

Reducing, we have $3x+170=491$; for adding 270 and subtracting 100, is the same as adding their difference.

6. A man aged 80 years, had spent a certain part of his life in France, three times as much and 30 years more in England, and twice as much wanting 10 years in America. How many years had he lived in each country?

7. A certain town contains 2900 inhabitants, English, Irish, and French; there are 600 fewer Irish than English, and 400 fewer French than Irish. How many are there of each?

Let $x =$ the number of English;
then $x - 600 =$ the number of Irish,
and $x - 600 - 400 =$ the number of French.

Hence, $x + x - 600 + x - 600 - 400 = 2900$; by reducing we have $3x - 1600 = 2900$; for all three of the numbers affected by the sign —, are considered as separately subtracted, and it would evidently be the same thing to subtract the sum of them at once.

8. In a casket containing 390 coins, there is a certain number of eagles, 10 less than twice as many half-eagles, and 20 less than three times as many dollars as there are half-eagles. How many coins are there of each kind?

9. A merchant bought a certain number of yards of broadcloth at \$8 per yard, 6 less than three times as many yards of cassimere at \$4 per yard, and twice as much silk at \$1 per yard as there were yards of cassimere. The whole came to \$1264. How many yards of each kind did he buy?

10. A man, engaged in trade, gained, the first year, \$500; the second year he doubled what he then had; but the third year he lost \$2000, when it appeared that he had remaining \$3000. How much money had he at first?

Suppose $x =$ his money at first.
Then $x + 500 =$ his money at the end of the 1st year;
$2x + 1000 =$ his money at the end of the 2d year,
and $2x + 1000 - 2000 =$ his money at the end of the 3d year.
Hence, $2x + 1000 - 2000 = 3000$; or reducing,
$2x - 1000 = 3000$; for $2x + 1000 - 2000$, signifies that 1000 is added to $2x$, and from the sum 2000 is subtracted, which is the same thing as subtracting 1000.

11. An inheritance of \$92500 is to be divided among five heirs, A, B, C, D and E, in the following manner, viz: B is to have \$600 more than A; C twice as much as B, wanting \$400; D as much as A and B both, wanting \$300; and E \$500 more than A and D both. What is the share of each?

Suppose $x =$ the share of A.
Then $x + 600 =$ the share of B,
$2x + 1200 - 400 =$ the share of C,
$x + x + 600 - 300 =$ the share of D,
and $x + x + x + 600 - 300 + 500 =$ the share of E.

Hence, $x + x + 600 + 2x + 1200 - 400 + x + x + 600 - 300 + x + x + x + 600 - 300 + 500 = 92500$. Reducing, $9x + 2500 = 92500$; for the sum of the numbers affected by the sign $+$ is 3500, and the sum of those affected by the sign $-$ is 1000; but adding 3500 and subtracting 1000 is the same as adding 2500.

Had the sum of the numbers, affected by the sign $-$, been greater than that of the numbers, affected by the sign $+$, the difference of these two sums would have had the sign $-$.

In the above question, the labor would have been abridged, if the expressions for the several shares had been reduced, as far as possible, previous to forming the equation.

12. A drover has a certain number of oxen; three times as many cows, wanting 25; just as many calves as cows; and 100 more sheep than he has oxen and cows together. The number of the whole is 905; how many of each has he?

13. In a company of 140 persons, consisting of officers, merchants and students, there were 4 times as many merchants as students, wanting 25; and 5 more than 3 times as many officers as students. How many were there of each class?

SECTION III.

EQUATIONS OF THE FIRST DEGREE, IN WHICH BOTH KNOWN AND UNKNOWN TERMS MAY OCCUR IN EACH MEMBER.

Art. **10**. What number is that to which if 18 be added, the sum will be equal to four times the number itself?

Let x represent the number; then $x + 18 = 4x$, or $4x = x + 18$; as it is evidently indifferent which quantity is made the first member.

Now it is our object to make all the x's, or unknown quantities, stand in one member of the equation, and the known quantities in the other; and, for the sake of uniformity, we generally collect the unknown quantities into the first member.

In the equation $4x = x + 18$, by transposing the x from the second member to the first, that is, by subtracting x from both members, we have $4x - x = 18$, or reducing,

$$3x = 18, \text{ and}$$
$$x = 6, \text{ Ans.}$$

Or we might have taken the equation $x + 18 = 4x$;
by transposing the 18, we have $x = 4x - 18$; then,
by transposing the $4x$, we have $x - 4x = -18$;
reducing, $-3x = -18$.

Here both members are wholly minus, but by transposing both, we have $18 = 3x$, which is the same as

$$3x = 18; \text{ hence,}$$
$$x = 6.$$

The equation, $3x = 18$, might have been obtained from $-3x = -18$, merely by changing the signs to $+$.

In like manner, in the equation $3x - 5x = 20 - 46$, which reduced gives $-2x = -26$, we might change all the signs before reducing, which would give $-3x + 5x = -20 + 46$, or $2x = 26$ and $x = 13$.

Art. **11**. Hence, *the signs of all the terms in both members of an equation may be changed; for this is the same as transposing all these terms.*

This change of signs should be made, whenever the first member becomes minus; but the learner must recollect, that terms having no sign, are supposed to have $+$, and that he must change *all* the signs, otherwise great errors will ensue.

1. Says A to B, if to my age twice my age and 30 years more be added, the sum will be five times my age. How old is he?

Let $x =$ his age;
then $5x = x + 2x + 30$. Reducing the 2d member,
$5x = 3x + 30$; transposing $3x$,

$$5x - 3x = 30; \text{ reducing,}$$
$$2x = 30, \text{ and}$$
$$x = 15 \text{ years. Ans.}$$

2. A merchant sells two kinds of cloth, the finer at $2 a yard more than the coarser; 12 yards of the coarser come to as much as 8 yards of the finer. What is the price of each per yard?

3. Says A to B, four times my age is equal to five times yours, and the difference of our ages is 10 years. What is the age of each?

4. A man having a certain number of cows and the same number of sheep, bought 4 more cows and 16 more sheep; he then found that three times his number of cows was equal to twice his number of sheep. How many had he of each at first?

5. A father distributed a certain sum of money among his four sons. The third received 9d. more than the youngest; the second, 12d. more than the third; and the eldest, 18d. more than the second. The whole sum was 6d. more than seven times what the youngest received. How much had each, and what was the whole sum distributed?

6. A sum of money was to be divided among six poor persons, so that the second should have 3s., the third 2s., the fourth 5s., the fifth 7s., and the sixth 8s., less than the first. Now the sum divided was 7s. more than four times the share of the second. What did each receive?

7. A person bought two casks of beer, one of which held twice as much as the other; from the larger he drew out 20, and from the smaller 25 gallons; he then found that there remained in the larger 4 times as much as in the smaller. What did each cask contain at first?

8. A man bought 10 bushels of wheat and 16 bushels of rye; the wheat cost 2s. more per bushel than the rye, and the whole cost of the wheat wanted 16s. to be equal to that of the rye. What was the price of each per bushel?

9. An instructor, wishing to arrange his pupils in rows with a certain number in each row, found that there were 3 too many to make six rows, and 4 too few to make seven rows. How

many did he wish to place in a row, and how many scholars had he?

Let $x =$ the number in each row;
then, $6x + 3 =$ the whole number of scholars;
also, $7x - 4 =$ the whole number of scholars.
Hence, $7x - 4 = 6x + 3$. (ax. 7).

10. A boy being sent to buy a certain number of pounds of meat, found, that if he bought pork, which was 9 cents per pound, he would have 5 cents left, but if he bought beef, which was 10 cents per pound, he would want 5 cents. How many pounds was he to buy, and how much money had he?

11. Two workmen received equal wages per day; but if the first had received 2s. more, and the second 2s. less per day, the first would have earned in 8 days as much as the second would in 12. What were the daily wages of each?

12. A and B began trade with equal stocks. In the first year A gained a sum equal to his stock and $27 over; B gained a sum equal to his stock and $153 over. The amount of both their gains was equal to five times the stock each had at first. What was the stock with which each began?

13. A man is 40 years old, and his son 9; in how many years will the father be only twice as old as the son?

14. A father is 66 years old and his son 30; how many years ago was the father three times as old as his son?

15. A grazier had two flocks of sheep, each containing the same number; from one of these he sold 50, and from the other 100, and found twice as many remaining in the one as in the other. How many did each flock originally contain?

16. A courier, who traveled 80 miles a day, had been gone one day, when another was sent from the same place to overtake him. In what time will the second, by traveling 90 miles per day, overtake the first, and at what distance from the starting-place?

17. A gentleman bought a horse and chaise; for the chaise he gave $75 more than for the horse, and three times the price

of the horse, diminished by \$50, was equal to twice the price of the chaise. Required the price of each.

18. When wheat was worth 5s. a bushel more than oats, a farmer gave 8 bushels of oats and 8s. in money for 4 bushels of wheat. What were wheat and oats worth per bushel?

19. A merchant, engaging in trade, during the first year doubled his stock, wanting \$500; the second year he doubled the stock he then had, wanting \$500; and so continued to double his stock each year, wanting \$500; until, at the end of the fourth year, he found he had \$500 more than eight times the stock with which he commenced. What was his stock at first?

20. Four towns are situated in the order of the four letters, A, B, C and D, and in the same straight line. The distance from B to C is 10 miles less than twice the distance from A to B; and the distance from C to D is 20 miles more than that from B to C; moreover, the distance from A to B, added to that from B to C, is equal to the distance from C to D and 5 miles more. What is the whole distance from A to D?

SECTION IV.

EQUATIONS OF THE FIRST DEGREE, CONTAINING FRACTIONAL PARTS OF SINGLE TERMS.

Art. **12**. 1. A merchant sold a bag of coffee for \$16, which was only four fifths of what it cost him. How much did it cost?

Let $x =$ the number of dollars it cost.

Then four fifths of x may be written $\frac{4}{5}x$, or more properly $\frac{4x}{5}$, which may be read either four fifths of x, four x fifths, one fifth of four x, or four x divided by five, the last of which is preferable.

Hence, $\frac{4x}{5} = 16$. Dividing both members by 4, we have

$\frac{x}{5} = 4$; if one fifth of x is equal to 4, the whole of x will be five times as much, or, $x = \$20$, Ans.

Or, we might first multiply by 5, and since a fraction is multiplied by dividing its denominator, we have $\frac{4x}{1}$ or $4x = 80$, and $x = 20$, as before. This latter method is generally preferable to the former.

2. A man said that one half and one fourth of his money amounted to \$75. How much money had he?

Let $x =$ his money.

Then $\frac{x}{2} + \frac{x}{4} = 75$. Multiplying by 2,

$$x + \frac{x}{2} = 150; \text{ multiplying this by 2,}$$

$$2x + x = 300; \text{ reducing,}$$

$$3x = 300; \text{ dividing both members by 3,}$$

$$x = \$100. \text{ Ans.}$$

3. In a certain school, one half of the boys learn Arithmetic; one fourth, French; one eighth, Grammar; one sixteenth, Algebra; and 10, Geometry. These classes constitute the whole school. How many boys does the school contain?

Suppose $x =$ the whole number of scholars.

Then, $x = \frac{x}{2} + \frac{x}{4} + \frac{x}{8} + \frac{x}{16} + 10$. (ax. 9). Multiply by 2,

$$2x = x + \frac{x}{2} + \frac{x}{4} + \frac{x}{8} + 20; \text{ multiply by 2,}$$

$$4x = 2x + x + \frac{x}{2} + \frac{x}{4} + 40; \text{ multiply by 2,}$$

$$8x = 4x + 2x + x + \frac{x}{2} + 80; \text{ multiply by 2,}$$

$$16x = 8x + 4x + 2x + x + 160; \text{ reducing,}$$

$$16x = 15x + 160; \text{ transposing } 15x \text{ and reducing,}$$

$$x = 160. \text{ Ans.}$$

Remark. Although it is generally safest to multiply by the denominators separately, we might, in this question, have multiplied the first equation by 16, the least common multiple of the

denominators; or we might have reduced all the unknown terms to a common denominator, which would have given $\frac{16x}{16} = \frac{15x}{16} + 10$, and $\frac{x}{16} = 10$, consequently $x = 160$.

4. A man found that he had spent one third of his life in Germany, one fourth in France, two fifths in England, and one year in the United States. How old was he, and how many years had he spent in the first three countries mentioned?

5. A merchant, on settling his affairs, found that he owed to one man $\frac{1}{8}$, to another $\frac{1}{7}$, and to a third $\frac{2}{15}$ of the money he had on hand; and that, after paying them, he should have \$3018 left. How much money had he, and how much did he owe each of the three creditors?

6. A goldsmith wished to make a mixture of gold, silver and copper, so that 2 ounces more than one third of the whole should be gold, 8 ounces more than one fourth of the whole, silver, and 2 ounces less than one sixth of the whole, copper. How many ounces in the whole mixture, and how many of each kind of metal?

7. A man left his estate to be divided between his wife and his three sons, in the following manner, viz: the wife was to have \$1000 less than one third of the whole estate; the eldest son, \$2000 more than one fifth of the whole; the second son, \$2000 more than one sixth of the whole; and the youngest son, exactly one sixth of the whole. What was the whole estate, and what were the portions of the several heirs?

8. A gentleman had spent 4 years more than one fourth of his life with his parents and at school, 12 years less than three fifths of it in the study and practice of his profession, and had lived in retirement 20 years. How old was he?

9. A's age is to B's as 4 to 3, and if twice B's age be added to A's, the sum will be 100 years. Required the age of each.

The meaning of the first condition is, that A's age is $\frac{4}{3}$ of B's, or that B's is $\frac{3}{4}$ of A's.

10. What is the length of a fish, whose head is 3 inches long,

his tail $\frac{1}{8}$ the length of his body, and his body as long as his head and tail?

Let $x =$ the length of the body.

11. Three fourths of a certain number exceeds five ninths of it by 14. What is that number?

12. A person, having spent one half of his money and one third of the remainder, had \$50 left. How much had he at first?

13. Says B to C, lend me \$200; C replies, I have not \$200 on hand, but if I had as much more and half as much more as I now have, and \$12½, I should have \$200. How much had he?

14. Divide 60 cents among three boys, so that the second shall have half as many as the first, and the third 10 more than one third as many as the second.

15. A man wished to distribute a certain number of apples amongst his four children, in such a manner, that the first should have one third of the whole; the second, three fifths as many as the first; the third, two thirds as many as the second; and the fourth, half as many as the third and 8 apples more. What was the whole number, and how many would each child receive?

16. A gentleman bought two horses and a chaise; the second horse cost once and a half as much as the first; and the chaise cost three times as much as the first horse; moreover the price of both horses wanted \$50 to be equal to that of the chaise. What was the cost of each horse and of the chaise?

17. A man found, that he expended one third of his yearly income for board, one eighth of it for clothes, and one twelfth of it for other purposes; and, that he had remaining \$550. What was his income, and what were his whole expenses?

18. A drover, having a certain number of sheep, sold one third of them and then bought 60, when he found he had twice as many as he had at first. What was his first number of sheep?

19. A gentleman gave to three persons £98. The second received five eighths of the sum given to the first, and the third, one fifth as much as the second. What did each receive?

20. A person set out on a journey, and went one seventh of the whole distance the first day, one fifth the second, one fourth the third, and 114 miles the fourth, at which time he completed his journey. How many miles did he travel in all, and how many each of the first three days?

SECTION V.

EQUATIONS OF THE FIRST DEGREE, CONTAINING FRACTIONAL PARTS OF QUANTITIES CONSISTING OF SEVERAL TERMS.

Art. **13.** 1. A says to B, I am 6 years older than you, and two thirds of my age is equal to three fourths of yours. What is the age of each?

Let $\quad x = $ B's age;
then, $x + 6 = $ A's age.

According to the conditions of the question, three fourths of the former must be equal to two thirds of the latter. One third of $x + 6$ is written $\frac{x+6}{3}$, and two thirds will be twice as much, or $\frac{2x+12}{3}$. Hence, $\frac{3x}{4} = \frac{2x+12}{3}$. Multiplying by 4,

$$3x = \frac{8x+48}{3}; \text{ multiplying this by 3,}$$

$$9x = 8x + 48; \text{ transposing and reducing,}$$

$$x = 48 \text{ years, B's age.} \quad x + 6 = 54 \text{ years, A's age.}$$

Remark. The division of a quantity consisting of several terms, as $x + 6$, is represented by placing the divisor under the dividend, care being taken to extend the line of separation under all the terms of the quantity to be divided.

2. A man bought a horse and saddle; for the horse he gave $230 more than for the saddle; and five times the price of the saddle was equal to two fifths of the price of the horse. Required the price of each.

Let $x =$ the price of the saddle;
then, $x + 230 =$ the price of the horse.
Hence, according to the conditions of the question,

$$5x = \frac{2x + 460}{5}.$$ Multiplying by 5,

$25x = 2x + 460$; transposing and reducing,
$23x = 460$; dividing by 23,
$x = \$20$, price of the saddle,
$x + 230 = \$250$, price of the horse.

3. A father's age is to that of his son as 5 to 2, and the difference of their ages is 30 years. Required their ages.

The first condition signifies that the father's age is $\frac{5}{2}$ of the son's, or that the son's is $\frac{2}{5}$ of the father's, or that 5 times the son's is equal to twice the father's.

Suppose $x =$ the age of the father;
then, $x - 30 =$ the age of the son.

Hence, $$x = \frac{5x - 150}{2}.$$ Multiplying by 2,

$2x = 5x - 150$; transposing and reducing,
$-3x = -150$; changing the signs,
$3x = 150$; dividing by 3,
$x = 50$ years, father's age;
$x - 30 = 20$ years, son's age.

4. A and B traded together. A put in \$100 more than B. The whole stock was to what A put in as 5 to 3. How much did each invest in trade?

5. A man's age, when he was married, was to that of his wife as 4 to 3; but after they had been married 10 years, his age was to hers as 5 to 4. How old was each at the time of their marriage?

6. A man's age, at the time of his marriage, was to that of his wife as 10 to 9; but if they had been married 10 years sooner, his age would have been to hers as 8 to 7. What were their respective ages at the time of marriage?

7. A and B have equal sums of money; but if B gives A

£10, $\frac{1}{2}$ of what A then has, will be equal to $\frac{3}{4}$ of what B has left. How much money has each?

8. Three towns are situated on the same straight road, and in the order of the letters A, B, C. The distance from B to C is 20 miles more than the distance from A to B, and is equal to $\frac{3}{5}$ of the whole distance from A to C. What are the distances from A to B, from B to C, and from A to C?

9. A merchant sold three packages of cloth; the second contained 15, and the third 30 yards more than the first; moreover, the third contained $\frac{4}{5}$ as much as the first two. How many yards were there in each?

10. A and B commence trade with equal stocks; A gains £10 per year, and B loses £5 per year; at the end of three years B has only $\frac{3}{4}$ as much property as A. How much has each at first?

11. Two boys, standing with bows and arrows on the bank of a river, undertook to shoot across it; the arrow of the first boy fell 10 yards short of the opposite bank, and that of the second fell 10 yards beyond it; now it was found that the first boy shot only $\frac{9}{11}$ as far as the second. What was the breadth of the river?

12. Two men have equal sums of money, but if one gives the other \$40, the former will have only $\frac{5}{9}$ as much as the latter. How much has each?

13. A, B and C counting their money, it was found that B had \$50 more than A and \$75 less than C, and that the sum of what A and B had, was $\frac{6}{7}$ of the sum of what B and C had. How much money had each?

14. A farmer, having a certain number of cows and twice as many sheep, sold 15 cows and bought 5 sheep; he then found that the number of cows was to the number of sheep as 3 to 13. How many of each had he at first?

15. A man engaged to work a year for \$200 and a suit of clothes; but falling sick, he worked only 5 months, and received \$60 and the suit of clothes. What was the value of the suit of clothes?

16. A man engaging in trade, gained the first year \$500, but the second year he lost $\frac{1}{5}$ of what he then had; after which he found that his stock was to that with which he began as 6 to 5. What was the stock with which he commenced?

17. A grocer bought 6 barrels of cider, and 7 barrels of beer; he gave \$2 a barrel more for the beer than for the cider; and $\frac{2}{4}$ of the price of the cider was equal to $\frac{3}{7}$ of the price of the beer. What was the price of each per barrel?

18. Two numbers are to each other as 9 to 10; but if 6 be added to each, the sums will be as 10 to 11. What are these numbers?

19. A man built two pieces of wall, one of which was 20 rods longer than the other; for the shorter he was to have \$3 a rod, and for the longer \$4 a rod; now the whole price of the former was to that of the latter as 3 to 8. What was the length of each piece?

20. A gentleman has two horses and one chaise; now if the first horse, which is worth \$100, be harnessed, he, with the chaise, will be twice the value of the second horse; but if the second horse be harnessed, he, with the chaise, will be four times the value of the first horse. What is the value of the chaise and of the second horse?

21. A man bought a horse, and afterwards paid \$50 for keeping him; he then sold him for $\frac{4}{5}$ of what he had already cost including the keeping, and received for him \$20 more than he first gave. How much did he pay for him at first?

22. Two cars run on different rail-roads; the speed of the second is 2 miles an hour greater than that of the first; and the distance passed over by the first in 8 hours, is $\frac{4}{5}$ of that passed over by the second in 9 hours. What is the speed of each per hour?

23. A gentleman started on a journey with a certain sum of money; after having had \$60 stolen from him, he expended one third of what he had left, and found that the remaining two thirds wanted \$90 to be equal to the sum which he carried from home. How much money had he on commencing his journey?

24. A man having a gold watch, paid $10 for repairing it, and then exchanged it for two silver watches of equal value, and after paying $5 for repairing one of these, he found that it had cost him $65. What was the value of the gold watch at first?

25. A shepherd, in time of war, was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what remained and $\frac{1}{2}$ of a sheep; after which he had but 34 sheep left. How many had he at first?

SECTION VI.

EQUATIONS OF THE FIRST DEGREE, WHICH REQUIRE THE SUBTRACTION OF QUANTITIES CONTAINING NEGATIVE TERMS.

Art. **14.** 1. A and B commenced business, A with twice as much money as B; A gained £20 and B lost £10; then the difference between A's and B's money was £70. How much did each begin with?

Suppose we knew, that B had £40 and A £80, when they began. Then, after A had gained £20, he would have $80+20$, or £100; and B having lost £10, would have left $40-10$ or £30; now to find the difference, we must subtract 30 from 100, which leaves 70. But, as in algebra most of the operations can only be represented, let us see how we can represent the preceding subtraction. Instead of 100 put its equivalent $80+20$, and instead of 30, its equivalent $40-10$; our object is to subtract the latter from the former. If we subtract 40 from $80+20$, it will be represented thus, $80+20-40$, which is the same as 60; but we wished to subtract only 30 or $40-10$; we have therefore subtracted too much by 10, and the remainder is too small by 10, consequently 10 must be added to $80+20-40$, which then becomes $80+20-40+10$ or 70. Hence we see, that, to subtract $40-10$, we must change the $+40$ to -40 and the -10 to $+10$.

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Now to solve the question; let $x =$ B's money;
then $2x =$ A's money.

When A had gained £20, he would have $2x + 20$;
and B having lost £10, he would have $x - 10$.
Subtracting B's from A's gives $2x + 20 - x + 10$.
For $x - 10$ is less than x by 10, and if we subtract x, we subtract too much by 10; the remainder then, after x has been subtracted, being 10 too small, 10 must be added to correct it.

Hence, $2x + 20 - x + 10 = 70$. Reducing the first member, $x + 30 = 70$; transposing and reducing,
$x =$ £40, B's money; $2x =$ £80, A's money.

2. Divide 40 into two parts such, that if three times the less be subtracted from twice the greater, the remainder will be 5.

Suppose $x =$ the greater part;
then $40 - x =$ the less part.

For if the greater were any known number as 30, for example, the less would be the remainder of 40, which is $40 - 30$ or 10; or if the greater were 28, the less would be $40 - 28$ or 12. So when the greater is represented by x, the less will be $40 - x$. Twice the greater is $2x$, and three times the less is $120 - 3x$, which, being subtracted from $2x$, gives $2x - 120 + 3x$.

Hence, $2x - 120 + 3x = 5$. Transposing and reducing.
$5x = 125$; hence, $x = 25$, the greater part,
and $40 - 25 = 15$, the less.

Art. **15**. It follows from the preceding questions and explanations, *that any quantity is subtracted by changing the signs of all its terms, and writing it after the quantity from which it is to be subtracted.*

1. A man has a horse and chaise, which together are worth \$400. Now if the value of the chaise be subtracted from twice that of the horse, the remainder will be the same, as if three times the value of the horse be subtracted from twice that of the chaise. Required the value of each.

2. A vintner has two equal casks full of wine; he draws 20 gallons out of one and 40 out of the other, and finds the differ-

ence between the number of gallons remaining in the two casks equal to one fourth of what each cask contained at first. How many gallons does each cask hold?

3. Divide the number 60 into two parts, such that the greater subtracted from 50, shall be equal to three times the less subtracted from 90.

4. A poulterer had a certain number of geese and twice as many turkeys; after having sold 10 geese and bought 30 turkeys, he found that if he subtracted $\frac{3}{5}$ of his number of geese from his number of turkeys, the remainder would be the same, as if he subtracted $\frac{8}{15}$ of his number of turkeys from four times his number of geese. How many of each had he at first?

Let $x =$ the number of geese;
then $2x =$ the number of turkeys.

After selling 10 geese and buying 30 turkeys, he would have $x - 10$ geese and $2x + 30$ turkeys. Then, according to the conditions of the question,

$$2x + 30 - \frac{3x - 30}{5} = 4x - 40 - \frac{16x + 240}{15}.$$ Multiplying by 5,

$$10x + 150 - 3x + 30 = 20x - 200 - \frac{16x + 240}{3};$$ multiplying by 3,

$$30x + 450 - 9x + 90 = 60x - 600 - 16x - 240;$$ transposing and reducing,

$-23x = -1380$; changing the signs,
$23x = 1380$; dividing by 23,
$x = 60$, the number of geese; and
$2x = 120$, the number of turkeys.

Observe that, after the equation was formed, in multiplying by 5, $3x$ was changed to $-3x$, and -30 to $+30$; for, the sign $-$ preceding the fraction, belongs to the whole fraction, and not to any particular part of it, and when the fraction is multiplied by 5, the numerator is to be subtracted; consequently $3x$, which is supposed to have the sign $+$, must be changed to $-3x$, and -30 to $+30$.

Also, in multiplying by 3, both the terms $16x$ and 240, being affected by the sign $+$, must receive the sign $-$. If, however, these fractions had been preceded by the sign $+$, the signs of the numerators would have remained unchanged. The same remarks are applicable to all similar cases.

Care must be taken also, when fractions are preceded by the signs $+$ and $-$, to make these signs stand even with the lines separating the numerators and denominators.

5. A farmer has 60 tons of hay; of this he sells a certain portion, and finds that $\frac{1}{2}$ of what he sells subtracted from $\frac{3}{4}$ of what he retains, gives the same remainder, as $\frac{3}{4}$ of what he retains subtracted from $\frac{5}{2}$ of what he sells. How many tons does he sell?

6. Two men, A and B, set out on a journey, each with the same sum of money. A spends \$40, and B \$30; then $\frac{3}{8}$ of A's money subtracted from $\frac{5}{7}$ of B's, would give $\frac{1}{4}$ of what each carried from home. How much money had each on commencing the journey?

7. Divide 147 into two parts, so that $\frac{1}{4}$ of the less subtracted from the greater, shall be equal to $\frac{1}{5}$ of the greater subtracted from the less.

8. A vintner had two casks of wine, each containing the same quantity; from the first he drew 10 and from the second 40 gallons; he then drew from the first $\frac{1}{3}$ as many gallons as the second contained after the first draught, and from the second $\frac{1}{3}$ as many gallons as the first contained after the first draught, and found the number of gallons remaining in the first cask to the number remaining in the second as 7 to 3. How many gallons did each cask hold?

9. A market woman having a certain number of eggs, sold 30 of them, and found that $\frac{2}{3}$ of what she had left, subtracted from what she had at first, would leave $\frac{5}{9}$ of what she had at first. How many had she before she sold any?

10. A man having a lease for 100 years, on being asked how much of it had already transpired, answered, that $\frac{3}{4}$ of the time past, subtracted from $\frac{5}{6}$ of the time to come, would leave the same remainder, as if $\frac{1}{15}$ of the time to come were subtracted

from $\frac{3}{5}$ of the time past. How many years had already transpired?

11. There is a pole consisting of three parts; the middle part is 4 feet longer than the lower and 4 feet shorter than the upper part; moreover, if $\frac{3}{10}$ of the upper part be subtracted from $\frac{2}{3}$ of the lower part, the remainder will be the same, as if $\frac{7}{8}$ of the middle part be subtracted from $\frac{4}{5}$ of the upper part. What is the length of each part and of the whole pole?

12. If a certain number be successively subtracted from 36 and 52, then $\frac{7}{8}$ of the first remainder be taken from $\frac{3}{4}$ of the second, the last remainder will be 10. What is that number?

SECTION VII.

MULTIPLICATION OF MONOMIALS.

Art. **16**. It may be remarked, that the addition, subtraction, multiplication and division of algebraical quantities, cannot, strictly speaking, be actually performed, in the same sense as they are in arithmetic, but are, in general, merely represented; these representations, however, are called by the same names as the actual operations in arithmetic.

A *monomial*, or *simple quantity*, consists of only one term (Art. **4**); as a, $b\,c\,m$, or $\frac{m}{n}$.

A *binomial* is a quantity consisting of two terms, as $a+b$, $am - xy$, or $\frac{a}{b}+cm$.

A *trinomial* is a quantity consisting of three terms, as $a+b - cd$.

Polynomial is a general name for any quantity consisting of several terms.

Moreover, any quantity containing more than one term, is called a *compound quantity*.

Art. **17**. The product of two simple quantities, such as a and b, is expressed, either by writing them with the sign of multiplication between them, as $a \times b$ or $a \,.\, b$, or by writing them after each other without any sign, as $a\,b$. The last form is most usual.

It is evidently immaterial in what order the letters are written; for, suppose $a=5$, and $b=7$; 5×7 is the same as 7×5; hence, the product of a by b may be written either $a\,b$ or $b\,a$.

In like manner, the product of a, b and c may be written $a\,b\,c$, $a\,c\,b$, $b\,a\,c$, $b\,c\,a$, $c\,b\,a$ or $c\,a\,b$. It is most convenient however to write them in the order of the alphabet.

The product of $5\,a\,b$ by $2\,c\,d$ might be written $5\,a\,b\,2\,c\,d$; but, as the order of the factors is unimportant, we may place the numerical factors next to each other, thus, $5 \times 2\,a\,b\,c\,d$, or performing the multiplication of 5 by 2, we have $10\,a\,b\,c\,d$. But we could not write $5\,2\,a\,b\,c\,d$, as the product of $5\,a\,b$ and $2\,c\,d$, because the value of a figure varies according to its place. If we wish to represent the multiplication of the figures, we must separate them either by letters, as $5\,a\,b\,2\,c\,d$, or by the sign of multiplication, as $5 \times 2\,a\,b\,c\,d$ or $5 \,.\, 2\,a\,b\,c\,d$.

The same result, $10\,a\,b\,c\,d$, may be obtained by another course of reasoning; d times $5\,a\,b$ is $5\,a\,b\,d$, $c\,d$ times $5\,a\,b$ is c times as much, or $5\,a\,b\,c\,d$, and $2\,c\,d$ times $5\,a\,b$ is twice as much as this last, or $10\,a\,b\,c\,d$.

By a similar course of reasoning, $5\,a\,c$, $4\,b\,d$ and $3\,m\,n$, multiplied together, would produce $60\,a\,b\,c\,d\,m\,n$.

We see from the preceding examples, that the product of two or more simple quantities, must consist of the product of the coëfficients, and all the letters of the several quantities.

1. Multiply $2\,a\,m$ by $7\,b\,c$.
2. Multiply 65 by $13\,c$.
3. Multiply $3\,x\,y$ by $7\,a\,b$.
4. Multiply $4\,a\,r$ by $6\,p\,q$.
5. Multiply $3\,g\,h$ by $17\,a\,x$.
6. Multiply $13\,x\,y$ by $12\,a$.
7. Multiply $4\,p\,q$ by $m\,n$.
8. Multiply $10\,p$ by $2\,a\,m$.
9. Multiply $7\,q$ by $3\,m\,s$.
10. Multiply $45\,x$ by $2\,a\,q$.
11. Multiply $a\,a$ by $a\,a\,a$.

The product in the last question would be, according to the principles given above, $a\,a\,a\,a\,a$. But when the same letter enters into a quantity several times as a factor, instead of writing that letter so many times, we may write it once only, and place a figure a little elevated at the right of it, to show the number of times it is contained as a factor. Thus, $a\,a$ is written a^2; $a\,a\,a$, a^3, and $a\,a\,a\,a\,a$, a^5.

A product must always contain all the factors both of the multiplicand and multiplier. In the present case, the letter a is twice a factor in the multiplicand, and three times in the multiplier; therefore, it must be contained five times as a factor in the product; that is, the product of a^2 and a^3 is a^5.

In like manner, the product of $3\,a^2\,b^3$ and $4\,a\,b^2$ is $12\,a^3\,b^5$; for, each letter must be contained as a factor in the product, as many times as it is in both multiplicand and multiplier. Also the product of $4\,a\,b$, $3\,a\,b^2$, and $2\,a^3\,b\,m$, is $24\,a^5\,b^4\,m$.

Art. **18**. This figure, placed at the right of a letter, is called the *index* or *exponent* of that letter, and affects no letter except that after which it is immediately placed. *An exponent then shows how many times a letter is a factor in any quantity.*

Letters written with exponents are called *powers* of those letters; thus, a^2 is called the *second power* of a, a^3 the *third power*, a^4 the *fourth power*, &c.; and, for the sake of uniformity, a, which is the same as a^1, is called the *first power* of a. When a quantity is written without any exponent, it is supposed to have 1 for its exponent.

In some treatises a^2 is called the *square*, and a^3 the *cube* of a; but these names belong to geometry rather than algebra. The words, square and cube, however, have the advantage of conciseness, and will occasionally be used in this work.

Figures also may be written with exponents; thus,

$$2^1 = 2.$$
$$2^2 \text{ or } 2\,.\,2 = 4.$$
$$2^3 \text{ or } 2\,.\,2\,.\,2 = 8.$$
$$2^4 \text{ or } 2\,.\,2\,.\,2\,.\,2 = 16.$$

The expression $a^3 b^2$, if 2 be put instead of a, and 3 instead of b, becomes $2^3 . 3^2$ or $2 . 2 . 2 . 3 . 3 = 72$.

Exponents must be carefully distinguished from coëfficients; for, $3a$ and a^3 have very different significations. Suppose $a = 10$; then $3a$ would be $3 . 10$ or 30, but a^3 would be $10 . 10 . 10$ or 1000.

From the preceding examples and observations, we derive the following

RULE FOR THE MULTIPLICATION OF MONOMIALS.

Art. **19**. *Multiply the coëfficients together, and write after their product all the different letters of the several quantities, giving to each an exponent equal to the sum of its exponents in all the quantities.*

1. Multiply $a^2 b^2$ by $7 a^3 b^4$.

In this question, the coëfficient of the multiplicand is 1, and that of the multiplier 7, the product of which is $7 . 1$ or 7; the sum of the exponents of a is 5, and the sum of the exponents of b is 6; hence, the answer is $7 a^5 b^6$.

2. Multiply $9 b c$ by $6 a b$.
3. Multiply $7 a^3 b^6$ by $43 a$.
4. Multiply $2 a^2 m^2 x$ by $23 a^5 m^5$.
5. Multiply $21 a$ by $19 a m x$.
6. Multiply $11 x^3 y^4$ by $x^3 y^4$.
7. Multiply $6 a b c d$ by $12 a^2 b^2 c^2 d^2$.
8. Multiply $73 m^4 x^4$ by $2 m^5 n^5$.
9. Multiply $5 a^2 b^3 c^4 d$ by $25 a^6 b^4 c^2 d$.
10. Multiply $9 p q x^2 y^2$ by $18 p^3 q x^5 y^7$.
11. Multiply $2 h^3 n^3 x$ by $6 a^6 h^3 n^5 x^6$.
12. Multiply $63 a r t^3$ by $9 a^2 r t^4 y$.
13. Multiply $11 x^2 y^2$ by $12 x^4 y p^3$.
14. Multiply $12 a m n p$ by $3 a^2 p^4 n m^5$.
15. Multiply $13 x^2 y^2 z$ by $3 x y z^3$.
16. Multiply $170 a b^2 c$ by $a^4 x$.
17. Find the product of $10 a^3$, $3 a b^2$, and $7 a b c$.
18. Find the product of $3 c^2 x$, $2 x y$, and $9 p x^2 y^2$.

19. Find the product of $\frac{4}{3} x^2 y^2$, $6 a x y$, and $7 p x^3 y^4$.
20. Find the product of $\frac{3}{4} a^5 m^4$, $\frac{1}{3} a x^2 y^2$, and $12 a m x^2 y^5$.

SECTION VIII.

REDUCTION OF SIMILAR TERMS.

Art. **20**. When several monomials are connected together by the signs $+$ and $-$, they form a *polynomial* or *compound quantity*. But such polynomials can frequently be reduced to a smaller number of terms. This can be done when some of the terms composing the polynomial, are similar.

Art. **21**. *Similar terms are those which are entirely alike with regard to the letters and exponents.*

N. B. In determining the similarity of terms, no regard is to be paid to the numerical coëfficients, or to the order of the letters.

The terms $3 a b$ and $5 a b$ are similar; so are $6 a^2 b^3$ and $9 a^2 b^3$. But $2 a^3 b^2$ and $4 a^2 b^3$ are not similar, because the exponents of a, as well as those of b, are different in the two quantities.

Suppose we have the polynomial $6 a^2 b - 2 m n + 4 a^2 b + 6 m n$. Here $6 a^2 b$ and $4 a^2 b$ are similar, and it is evident that 6 times and 4 times the same quantity make 10 times that quantity; hence, $6 a^2 b + 4 a^2 b$ is $10 a^2 b$; also, $-2 m n + 6 m n$, or $6 m n - 2 m n$, is $+ 4 m n$; the given polynomial, therefore, becomes $10 a^2 b + 4 m n$.

Again, take the polynomial, $24 b^2 c^3 - 3 a b c^2 + 2 b^2 c^3 - 5 p q - 6 a b c^2 - 2 p q - 3 m^2 z$. Here, $24 b^2 c^3 + 2 b^2 c^3 = 26 b^2 c^3$; $-3 a b c^2 - 6 a b c^2 = -9 a b c^2$; and $-5 p q - 2 p q = -7 p q$; hence, the given quantity becomes $26 b^2 c^3 - 9 a b c^2 - 7 p q - 3 m^2 z$.

Suppose the following polynomial given, viz: $2 a^2 b c^2 - 4 a^3 b c + 12 a^2 b c^2 - 4 a^3 b c + 12 a^3 b c + 5 a^2 b c^2 + 3 a^3 b c - 10 a^2 b c^2 - a^2 b c^2$. First, collect the positive terms of one kind; $2 a^2 b c^2 + 12 a^2 b c^2 + 5 a^2 b c^2 = 19 a^2 b c^2$; then collect the

negative terms of the same kind; $-10\,a^2\,b\,c^2 - a^2\,b\,c^2 = -11\,a^2\,b\,c^2$; the five terms, therefore, are the same as $19\,a^2\,b\,c^2 - 11\,a^2\,b\,c^2$, or $8\,a^2\,b\,c^2$. In like manner, the positive and negative terms of the other kind being separately collected, give $15\,a^3\,b\,c - 8\,a^3\,b\,c$, or $7\,a^3\,b\,c$. The polynomial therefore becomes $8\,a^2\,b\,c^2 + 7\,a^3\,b\,c$.

Sometimes the sum of the negative terms exceeds that of the similar positive terms; in such cases, we must take the difference between the two sums and give it the sign $-$.

Suppose we have $m + 8\,a\,b^2 - 13\,a\,b^2 + 10\,a\,b^2 - 12\,a\,b^2$; collecting the similar positive and negative terms separately, we have $m + 18\,a\,b^2 - 25\,a\,b^2$; this is the same as $m + 18\,a\,b^2 - 18\,a\,b^2 - 7\,a\,b^2$. But $+18\,a\,b^2$ and $-18\,a\,b^2$ destroy each other, and the result is $m - 7\,a\,b^2$.

Art. **22**. Hence we derive the following

RULE FOR THE REDUCTION OF SIMILAR TERMS.

Unite all the similar terms of one kind affected with the sign $+$, *by adding their coëfficients and writing the sum before the common literal quantity; unite, in like manner, the similar terms of the same kind affected with the sign* $-$; *then take the difference between these two sums, and give the result the sign of the greater quantity.*

Remark. The learner will be less liable to error, if he take the precaution to mark the terms, as he reduces them.

Reduce the following polynomials to the least number of terms.

1. $6\,a^2\,b - 8\,a^2\,b - 9\,a^2\,b + 15\,a^2\,b - a\,b^2$.

2. $7\,a\,b\,c^2 - a\,b\,c^2 - 7\,a\,b\,c^2 - 8\,a\,b\,c^2 + 12\,a\,b\,c^2$.

3. $16\,a\,b\,c^3 + 5\,a\,b^3\,c + 7\,a^3\,b\,c - 10\,a\,b\,c^3 - 7\,a^3\,b\,c - 4\,m\,n - 4\,a\,b^3\,c - 6\,a\,b^3 + 2\,a\,b\,c^3$.

4. $12\,p^3\,c\,r + 16\,m^2\,n^2 - c + 4\,c\,p^3\,r + 3\,c + 7\,g\,h - 17\,p^3\,c\,r - 12\,m^2\,n^2 - 2\,c + 3\,m^2\,n^2$.

5. $6\,m\,n\,r + 11\,p^3\,q - 17\,m\,n\,r + 3\,b\,c - 22\,p^3\,q + 18\,m\,n\,r + 3\,p^3\,q - 5\,m\,n\,r + 7\,b\,c$.

6. $22\,x^2 - y^3 + 3\,x^2 + 8\,x^2 - 7\,x^2 + m + 7\,y^3 - 16\,y^3 + 3\,m + 7\,m + x^2 + 2\,y^3 - 6\,m$.

7. $a^3pm + apm^3 + 6 + 3ap^2m + 7a^3pm - 4apm^3 + a^3mp + 6amp^2 - 8amp^2 - 12a^3mp + 10a^3mp$.

8. $abc^3 + 7mn^2p^3r^4 + 6abc^2 + 9ab^2c + 12m^2n + 2abc^3 + 2m^2n^2p^3r^4 - 7abc^2 + 2abc^2 - 10ab^2c - 6mn^2p^3r^4 + ab^2c + abc^2 - 3m^2n^2p^3r^4 - 3abc^3 - 2abc^2$.

SECTION IX.

ADDITION.

Art. **23**. The addition of *positive monomials* or *simple quantities*, consists merely in writing them after each other, and giving to each of them the sign $+$, except the first, which is also supposed to have the sign $+$. Thus, to express the addition of a and b, we write $a+b$ or $b+a$. Also, to signify the addition of a, b, c and d, we write $a+b+c+d$. In like manner, the addition of ab, $3xy$, and $4mn$, is expressed thus, $ab+3xy+4mn$. The order of the terms, as was observed in Art. **9**, is unimportant.

Art. **24**. If it were required to add together the *polynomials*, $a+b+c$, and $m+n$, in which all the terms are affected with the sign $+$, the process would evidently be the same, as if it were required to add together the separate terms of which these polynomials are composed; that is, we should write them after each other, giving the sign $+$ to every term except the first; and the sum would be $a+b+c+m+n$.

But if some of the terms in the polynomials to be added, have the sign $-$, they must retain the same sign in the sum. Take an example in arithmetical numbers. Let it be proposed to add $10-3$ to 12; $10-3$ is 7; we wish then to add 7 to 12. But, if we first add 10, which is expressed thus, $12+10$, the sum is too great by 3; therefore, after having added 10, we must subtract 3, and the true sum is $12+10-3$ or 19.

Let us now add $b-c$ to a. First add b to a, and we have $a+b$; this is too great by c, because the quantity $b-c$, which

was to be added, is less than b by the quantity c; therefore, after having added b, we must subtract c, and the true result is $a + b - c$.

We see, in these instances, that we have merely written the terms after each other without any change in their signs; and the reasoning used in explanation of the process, is applicable to the addition of all polynomials, in which some of the terms are affected with the sign $-$.

The sum of the polynomials $a^2 b + 3c - 4a$ and $12c + 8a^2 b - 3a$ is $a^2 b + 3c - 4a + 12c + 8a^2 b - 3a$. But this sum contains similar terms, which may be reduced, according to the principles given in Art. **22**. This reduction being made, the sum, in its simplified form, becomes $9a^2 b + 15c - 7a$.

Art. **25**. From what has been said above, we deduce the following

RULE FOR THE ADDITION OF ALGEBRAIC QUANTITIES.

Write the several quantities one after another, giving to each term its proper sign, and then reduce the similar terms.

Observe, that those terms which have no sign, are supposed to have the sign $+$

1. Add together $4a$, $6b$, $7c$, $9a$,
$3a + 6$, $6c$, $4d$,
and $4a + 3c - 4b$.
2. Add together $3ab - 4cd$, $m^2 n$,
$9ab + 8cd$, $3m^2 n - m$,
and $4m$.
3. Add together $3ab + 4cd - m^2$,
$4cm - 7cd + 3ab$,
$12ab + 8cd + 6m^2$.
4. Add together m^2,
$a^2 b^2 + 6m^2 - 6mn$,
$4a^2 b^2 - 12mn + 8m^2$,
$4xy - 7m^2 + 3b + 8a^2 b^2$.
5. Add together $11bc + 4ad - 8ac + 5cd$,
$8ac + 7bc - 2ad + 4mn$,

$2cd - 3ab + 5ac + an,$
$9an - 2bc - 2ad + 5cd.$

6\. Add together $5a + 4b - 3c,$
$8 - 7d,$
$3a - 12b,$
$7c - 10d - 4,$
$16 - 3c + 5.$

7\. Add together $3b - a - c - 115d,$
$6c - 5f - d,$
$3a - 2b - 3c + 27e,$
$3e - 7f + 5b - 8c,$
$17c - 6b - 7a,$
$11f - 5e + 9d + g - 3a,$
$6e - 5c - 2d - 9f.$

8\. Add together $4a^2b + 3c^3d - 9m^2n,$
$4m^2n + ab^2 + 5c^3d + 7a^2b,$
$6m^2n - 5c^3d + 4mn^2 - 8ab^2,$
$7mn^2 + 6c^3d - 5m^2n - 6a^2b,$
$7c^3d - 10ab - 8m^2n - 10d^4,$
$12a^2b - 6ab^2 + 2c^3d + mn.$

9\. Add together $2ab^2 + 3ac^2 - 8cx^2 + 9b^2x - 8hy^2,$
$5a^3 - 4ab^2 - 7bx^2 - b^2x - 4ky^2 - 15hy,$
$5ky - hy^2 + 11x + 14b^3 - 22ac^2 - 10x^2,$
$19ac^2 - 8b^2x + 9x^2 + 6hy + 2ky^2,$
$2ab^2 + 7py^2 - 10ky + 3a^3 + 2x.$

SECTION X.

SUBTRACTION.

Art. **26**. We have already seen, that a simple quantity is subtracted, by giving it the sign $-$; thus, to subtract b from a, we write $a - b$.

We are now to show how to subtract polynomials. If it were required to subtract $7 + 3$ from 12, it is evident that 7 and 3

must both be subtracted, which is expressed thus, $12-7-3$. In like manner, if $b+c$ is to be subtracted from a, b and c must both be subtracted, thus, $a-b-c$.

But if some of the terms in the polynomial to be subtracted, have the sign $-$, the signs of these terms must be changed to $+$. Suppose it were required to subtract $7-5$ from 10; $7-5$ is 2, and 2 from 10 leaves 8. Now if we subtract 7 from 10, which is represented thus, $10-7$, we subtract too much by 5, and the remainder 3 is too small by 5; consequently, after having subtracted 7, we must add 5, and the true result is $10-7+5$, or 8.

Now let us subtract $b-c$ from a. First subtract b, and we obtain $a-b$; but b is greater that $b-c$ by c; therefore, as we have subtracted too much by c, the remainder is too small by c; we must, consequently, add c to $a-b$, and we have $a-b+c$ for the true result.

The same reasoning is applicable to the subtraction of all polynomials containing negative terms.

Art. **27**. Hence, we deduce the following

RULE FOR THE SUBTRACTION OF ALGEBRAIC QUANTITIES.

Change the signs of all the terms in the quantity to be subtracted, and write it after that from which it is to be subtracted; then reduce the similar terms.

1. From $8a+4b$, subtract $3a-2b$.

Changing the signs of the latter quantity, and writing it after the former, we have $8a+4b-3a+2b$, which reduced, becomes $5a+6b$. Ans.

2. From $4ab-3bc$, subtract $2ab-6bc$. Ans. $2ab+3bc$.
3. From $4ab-3c^2+bc$, subtract $ab-c^2-2bc$.
4. From $5ac-8ab+9bc-4am$, subtract $8am-2ab+11ac-7cd$.
5. From $3m-8x-7f$, subtract $3d-5m-2x-6f+8$.

Art. **28**. Sometimes it is convenient to *represent* the subtraction of polynomials, without actually performing the operation,

This is done by enclosing the quantity within *brackets* or a *parenthesis*, and prefixing the sign —. Thus, $6ab - 3c + d - (3ab - 4c + 2d)$ signifies that $3ab - 4c + 2d$ is to be subtracted from $6ab - 3c + d$. Performing the subtraction indicated and reducing, remembering that $3ab$ within the parenthesis has the + sign, we obtain $3ab + c - d$.

According to this principle, a polynomial may be made to undergo various transformations.

For example, $3ab - a - b$ is the same as $3a - (a + b)$; for, when the subtraction, indicated in the latter expression, is performed, it becomes $3ab - a - b$.

In like manner, $7a^3 - 8a^2b - 4b^2c + 6b^3$ is equivalent to $7a^3 - 8a^2b - (4b^2c - 6b^3)$.

Let the learner perform the subtraction indicated in the following examples.

1. $27a^2x - 2bc + 4x^2 + 3ax - (9a^2x + 4bc - 6x^2 - b + 4ax + 6)$.
2. $28ax^3 - 16a^2x^2 + 25a^3x - 13a^4 - (18ax^3 + 20a^2x^2 - 24a^3x - 7a^4)$.
3. $30ab - 6bc^2 + 2b^2 - 45 - (4ab + 12bc^2 - 24b^2 - c^2 + m^2x - 92)$.
4. $8a^2xy - 5bx^2y + 17cxy^2 - 9y^5 - (a^2xy + 3bx^2y - 13cxy^2 + 20y^5)$.
5. $63x^2y^2 + 24xy + b^2c^2 - cd^3 + 5c^3d - (45x^2y^2 - 24xy - 3b^2c^2 - 2cd^3 + 4c^3d - m)$.

Art. **29**. The reverse of the process in Art. **28** is sometimes very useful, viz: putting within a parenthesis part of a polynomial, and placing the negative sign before the parenthesis. To do this, it is only necessary to change the signs of all the terms, placed within the parenthesis.

Thus, $a + b - c = a - (-b + c) = a - (c - b)$; also, $4abc - 6c^2d + m^2 - 7pq = 4abc - (6c^2d - m^2 + 7pq)$.

Let the learner throw the last two terms of each of the following quantities into a parenthesis, preceded by the negative sign.

1. $a m - b c + d m.$
2. $a b c - d^2 - 3 m y + x y.$
3. $a + b - c - d.$
4. $a^2 + b^2 + 2 a b - a - b.$
5. $4 m^2 + 12 m n + 9 n^3 - 2 m - 3 n.$
6. $4 a b + 7 a m - x^2 + y^2.$
7. $7 m n + x^2 - p q + y^3.$

SECTION XI.

MULTIPLICATION OF POLYNOMIALS.

Art. **30**. Multiply $10 + 3$ by 7. It is manifest, that 10 and 3 must both be multiplied by 7, and the products added.

Operation.

$$\begin{array}{l} 10 + 3 \\ \underline{\;7\qquad\qquad\qquad\qquad} \\ 70 + 21 = 91 = 13 \,.\, 7. \end{array}$$

So, to multiply $a + b$ by c, each of the terms, a and b, must be multiplied by c, and the products added.

Operation.

$$\begin{array}{l} a \;+\; b \\ \underline{c\qquad\quad} \\ a c + b c. \end{array}$$

If there are several terms in both multiplicand and multiplier, all the terms of the former must be multiplied by each term of the latter.

Multiply $8 + 3$ by $7 + 5$. Here we must take $8 + 3$ seven times and five times, and then add the products.

Operation.

$$\begin{array}{l} 8 + 3 \\ \underline{7 + 5\qquad\qquad\qquad\quad} \\ 56 + 21 + 40 + 15 = 132. \end{array}$$

If both quantities be reduced before multiplying, and then their product taken, the result will be the same; thus, $11 . 12 = 132$.

In like manner, if $a + b$ is to be multiplied by $c + d$, we must multiply $a + b$ by c and then by d, and take the sum of the products.

Operation.

$$\begin{array}{l} a + b \\ \underline{c + d \qquad\qquad\quad} \\ a c + b c + a d + b d. \end{array}$$

Let us suppose now that the multiplicand contains a negative term.

Multiply $10 - 6$ by 3. The multiplicand, $10 - 6$, is 4, and 3 times 4 is 12. But, if we multiply 10 by 3, the product 30 is too great by 3 times 6 or 18, which must therefore be subtracted from 30; the true product then is $30 - 18$ or 12.

Operation.

$$\begin{array}{l} 10 - 6 \\ \underline{3 \qquad\qquad\quad} \\ 30 - 18 = 12. \end{array}$$

So, to multiply $a - b$ by c; if we first multiply a by c, the product $a c$ is too great by c times b or $b c$, which must therefore be subtracted from $a c$; the true result then is $a c - b c$.

Operation.

$$\begin{array}{l} a - b \\ \underline{c \qquad\quad} \\ a c - b c. \end{array}$$

The term $- b c$ in the product, shows, that when a negative term, as $- b$, is multiplied by a positive term, as $+ c$, the product must be negative.

Now let both multiplicand and multiplier contain negative terms.

Multiply $18 - 3$ by $12 - 5$. If we reduce both numbers,

and then multiply them together, the product is $15 \cdot 7$ or 105. But to perform the operation without reducing, we first multiply $18-3$ by 12, which gives $216-36$; but, as we wished to take $18-3$ only $12-5$ or 7 times, we have taken it 5 times too many times; we must, therefore, subtract 5 times $18-3$ or $90-15$ from $216-36$, which gives $216-36-90+15$. See Art. **27**.

Operation.

$$\begin{array}{l} 18-3 \\ 12-5 \\ \hline 216-36-90+15=105. \end{array}$$

In a similar manner, to multiply $a-b$ by $c-d$, we first take c times $a-b$, which, as we have already seen, is $ac-bc$; but as we wished to take $a-b$ only $c-d$ times, we have taken it d times too many times. Now d times $a-b$, d being considered as positive, is $ad-bd$; this then must be subtracted from $ac-bc$, and we have $ac-bc-ad+bd$ for the true product of $a-b$ by $c-d$.

Operation

$$\begin{array}{l} a-b \\ c-d \\ \hline ac-bc-ad+bd. \end{array}$$

We see that the term $-ad$ is produced by multiplying $+a$ by $-d$; hence, if a positive term be multiplied by a negative, the product must be negative. Moreover, the term $+bd$ is produced by multiplying $-b$ by $-d$; therefore, if two negative terms are multiplied together, the product must be positive.

Art. **31**. From the preceding explanations, we derive the following

RULE FOR THE MULTIPLICATION OF POLYNOMIALS.

1. *Multiply all the terms of the multiplicand by each term of the multiplier separately, according to the rule for the multiplication of simple quantities.*

2. *With regard to the signs, observe, that if the two terms to be multiplied together, have the same sign, either both + or both —, the product must have the sign +; but if one term has the sign + and the other the sign —, the product must have the sign —.*

3. *Add together the several partial products, reducing terms which are similar.*

1. Multiply $2ab + bc + 3xy$

$$\begin{array}{l} 2ab + bc + 3xy \\ \text{by } \underline{3ab + 2bc} \end{array}$$

$$\left.\begin{array}{l} 6a^2b^2 + 3ab^2c + 9abxy \\ + 4ab^2c + 2b^2c^2 + 6bcxy \end{array}\right\} \text{Partial products.}$$

$$6a^2b^2 + 7ab^2c + 2b^2c^2 + 9abxy + 6bcxy;$$

which is the result reduced.

Remark. It is convenient, in order to facilitate the reduction, to place similar terms, in the partial products, under each other.

2. Multiply

$$4a^3 - 5a^2b - 8ab^2 + 2b^3, \text{ by}$$
$$2a^2 - 3ab - 4b^2$$

$$\left.\begin{array}{l} 8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3 \\ - 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4 \\ - 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5 \end{array}\right\} \text{Partial products.}$$

$8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5$. Result reduced.

3. Multiply

$$a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5, \text{ by}$$
$$a - b$$

$$\left.\begin{array}{l} a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 \\ - a^5b - a^4b^2 - a^3b^3 - a^2b^4 - ab^5 - b^6 \end{array}\right\} \text{Partial products.}$$

$a^6 - b^6$. Result reduced.

4. Multiply $2a - 3$ by $a + 4$.

5. Multiply $a^2 + 2ab + 2b^2$ by $a^2 - 2ab + 2b^2$.

6. Multiply $7ab - 3ac^2 + 4c^2d^2$ by $4ac^2 - 3ab - 2c^2d^2$.

7. Multiply $14a^3c - 6bc + 12xy^2 - c^2$ by $9a^3c + 6bc - 2x^2y + 3c^2$.

8. Multiply $3a^4b - 4a^3b^2 + 6a^2b^3 - ab^4$ by $a^2 - 2ab + b^2$.

9. Multiply $14b^3 + 28b^2c - 7bc^2 + c^3$ by $3b^2 - 6bc - 2c^2$.

Art. **32**. Sometimes the multiplication of polynomials is *indicated*, without being actually performed. This is done by placing a horizontal straight line, called a *vinculum*, over each of the polynomials, and writing them after each other with the sign of multiplication between them; or, by enclosing each in a parenthesis, and writing them after each other, either with or without the sign of multiplication. Thus, each of the expressions, $\overline{a+b} \times \overline{m-n}$, $\overline{a+b} \, . \, \overline{m-n}$, $(a+b) \times (m-n)$, $(a+b) \, . \, (m-n)$, and $(a+b)\,(m-n)$, represents the multication of $a+b$ by $m-n$.

The last mode of representation is generally preferred; but we must be careful to include each polynomial in a parenthesis; for, $(a+b)\,m - n$ indicates that $a+b$ is multiplied by m only, and that n is subtracted from the product.

In like manner, $(a+b)\,(m-n)\,(x+y)$ indicates that the first polynomial is multiplied by the second, and that product by the third.

Let the learner perform the operations indicated in the following examples.

1. $(a^2 + 2ab - c^2)\,(a - b)$.
2. $(a^2 + 2a^2b - 2b^2)\,(a^2 - 2a^2b + 2b^2)$.
3. $(m^3 + m^2n + mn^2 + n^3)\,(m - n)\,(m^4 + n^4)$.
4. $(a^2 + b^2 + c^2)\,(b + c) + (a^2 - b^2 - c^2)\,(b - c)$.

In the last example, the first polynomial is to be multiplied by the second, and the third by the fourth, and then the products are to be added.

5. $(3x^2 + 6xy + 3y^2)\,(x - y) + (4x^2 - 6xy - 3y^2) \times (x + y)$.
6. $(x - a - b)\,(4ac - 2) + (2x + 2a + 3b)\,(3ac + 2)$.
7. $(a^2 + 2ab + b^2)\,(c + d) - (a^2 - 2ab + b^2)\,(c - d)$.

In this last example, the product of the last two polynomials is to be subtracted from the product of the first two.

8. $(4x^2 - 4xy + c^2)(a+b) - (a^2 m + xy)(c^2 - d^2) \times (a+x)$.

9. $b^2(13bc^2x - 7c^3x^2) - c^2(b^2cx^2 - 18c^3x^2)$.

10. $(112 - 4x + a^2)(b+4) + a^2(10 + 6bc - 7c^2)$.

Art. **33**. The following cases of multiplication deserve particular attention, on account of the practical application, which will hereafter be made of the results.

Let a and b represent any two quantities; their sum is $a+b$, and their difference $a-b$. Multiply $a+b$ by $a-b$.

Operation.

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad\;\; -ab-b^2} \\ a^2-b^2. \end{array}$$

The product, $a^2 - b^2$, is the difference between the second power of a and the second power of b. Hence,

The sum of two quantities multiplied by their difference, gives the difference of the second powers of those quantities.

Suppose, for example, two numbers, 10 and 4; then, $(10+4) \times (10-4) = 100 - 16 = 84$.

In like manner, $(3a + 4b)(3a - 4b) = 9a^2 - 16b^2$.

Art. **34**. When a polynomial is multiplied by itself, the result is called the *second power* of that polynomial, and when it is multiplied twice by itself, the result is called the *third power*.

Find the second power of the binomial $a+b$.

Operation.

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad\;\; +ab+b^2} \\ a^2+2ab+b^2. \end{array}$$

Hence, the second power of the sum of two quantities, contains the second power of the first quantity, plus twice the product of the first by the second, plus the second power of the second.

Suppose the two numbers 12 and 3; then, $(12+3)(12+3) = 144+72+9 = 225$. In like manner, the second power of $3ab+2c$ or $(3ab+2c)(3ab+2c) = 9a^2b^2+12abc+4c^2$.

Art. **35**. Find the second power of $a-b$.

Operation.

$$\begin{array}{r} a-b \phantom{{}+b^2} \\ \underline{a-b \phantom{{}+b^2}} \\ a^2-ab \phantom{{}+b^2} \\ \underline{-ab+b^2} \\ a^2-2ab+b^2. \end{array}$$

This differs from the second power of $a+b$ only in the sign of $2ab$, twice the product of the two quantities, which is in this case minus.

Suppose the two numbers 8 and 2; then, $(8-2)(8-2) = 64-32+4 = 36$. In the same manner, $(6bc-2m) \times (6bc-2m) = 36b^2c^2-24bcm+4m^2$.

Art. **36**. Let the learner multiply the second power of $a+b$ by $a+b$; the result, that is the third power of $a+b$, is

$$a^3+3a^2b+3ab^2+b^3.$$

Hence, the third power of the sum of two quantities, contains the third power of the first quantity, plus three times the second power of the first into the second, plus three times the first into the second power of the second, plus the third power of the second.

Art. **37**. We find, in like manner, that the third power of $a-b$ is

$$a^3-3a^2b+3ab^2-b^3,$$

which differs from the third power of $a+b$ only in the signs of the second and fourth terms, which in this case are negative.

SECTION XII

DIVISION OF MONOMIALS OR SIMPLE QUANTITIES.

Art. **38**. In multiplication, two factors are given, and it is required to find the product; whereas, in division, one factor and the product, that is, the divisor and dividend, are given, and it is required to find the other factor or quotient.

Division then is the reverse of multiplication; and it is evident, that the divisor and quotient multiplied together, must reproduce the dividend. For this purpose, the coëfficient of the quotient must be such, as multiplied by the coëfficient of the divisor, will produce that of the dividend; and the exponent of any letter in the quotient, must be such, as added to its exponent in the divisor, will produce the exponent of that letter in the dividend.

Divide $a\,b$ by a, or find the $\frac{1}{a}$ part of $a\,b$.

The quotient is b, because the product of a and b is $a\,b$. Or if $a\,b$ be divided by a, the quotient is b, for the same reason.

Divide $3\,a\,b\,c$	by $a\,b$.	Ans. $3\,c$.
Divide $8\,b\,c\,x$	by $2\,x$.	Ans. $4\,b\,c$.
Divide a or $1\,a$	by a.	Ans. 1.
Divide a or $1\,a$	by 1.	Ans. a.
Divide $a\,b$	by $a\,b$.	Ans. 1.
Divide $a\,b\,c$	by 1.	Ans. $a\,b\,c$.
Divide $21\,x\,y$	by $7\,x$.	Ans. $3\,y$.
Divide $84\,a\,b\,c\,x$	by $12\,c$.	Ans. $7\,a\,b\,x$.

The above answers are correct, because in each case the quotient multiplied by the divisor, produces the dividend.

Divide a^7 by a^4. Ans. a^3; because $a^4 \,.\, a^3 = a^7$.

Divide $3\,a^3\,b^5$ by $a\,b^2$. Ans. $3\,a^2\,b^3$; because $3\,a^2\,b^3 \,.\, a\,b^2 = 3\,a^3\,b^5$.

Art. **39**. In the multiplication of monomials, when the same letter occurred in both factors, the exponents of that letter were added; and we see, from the last two examples, that, in division,

when the same letter occurs in both dividend and divisor, the exponent of that letter in the latter, must be subtracted from its exponent in the former.

If a be divided by a, the quotient is 1; for, any quantity divided by itself, gives unity for a quotient. But, if we perform the division by subtracting the exponents, we have $\frac{a}{a}$, or $\frac{a^1}{a^1} = a^0$. Hence, (ax. 7), $a^0 = 1$. *That is, any quantity with zero for an exponent is equal to* 1.

Art. **40.** From the preceding examples and observations, we derive the following

RULE FOR DIVIDING ONE MONOMIAL BY ANOTHER.

1. *Divide the coëfficient of the dividend by the coëfficient of the divisor.*

2. *Strike out from the dividend the letters common to it and the divisor, when they have the same exponents in both; but, if the exponents of any letter are different, subtract its exponent in the divisor from that in the dividend, and write the letter in the quotient with an exponent equal to the remainder.*

3. *Write also in the quotient, with their respective exponents, the letters of the dividend not found in the divisor.*

Remark. If in any case, however, the divisor and the dividend are equal, the quotient will be 1.

1. Divide $36\,a^4\,b^3\,c^2\,m$ by $4\,a^3\,b\,c^2$. Ans. $9\,a\,b^2\,m$.
2. Divide $48\,a^3\,b^3\,c^2\,d$ by $12\,a\,b^2\,c$.
3. Divide $150\,a^5\,b^8\,c\,d^3$ by $30\,a^3\,b^5\,d^2$.
4. Divide $120\,a\,b^7\,h^3$ by $10\,a\,b$.
5. Divide $125\,m^5\,x\,y^7$ by $5\,x\,y$.
6. Divide $93\,a^3\,b^3\,m^3\,x$ by $3\,a\,b\,m\,x$.
7. Divide $111\,m^4\,a^6\,b^7\,x\,y^3$ by $37\,m\,x\,y^2\,b^3$.
8. Divide $27\,a^2\,h^2\,n^2$ by 27.
9. Divide $15\,a\,m^2\,x$ by $a\,m^2\,x$.
10. Divide $27\,a^5\,m^9\,x^4\,y$ by $3\,a\,m^5\,x$.
11. Divide $729\,x^4\,y^4$ by $9\,x^3\,y$.
12. Divide $16\,a^5\,b^4\,x^2$ by $8\,a\,b^2\,x^2$.

13. Divide $99\,m\,x^5\,y^7\,z^3$ by $33\,x\,y^4\,z^2$.
14. Divide $1008\,a^5\,b^5\,c^2\,d^2\,x$ by $8\,a\,b\,c^2\,x$.
15. Divide $111\,m^5\,n^6\,p^7$ by $3\,m^4\,n^5$.
16. Divide $115\,n^3\,r^6\,s^2$ by $5\,n\,r\,s$.
17. Divide $75\,x^{10}\,y^{13}$ by $25\,x^8\,y^3$.
18. Divide $350\,a^7\,b^{14}\,x^7$ by $50\,a^7\,x^7$.
19. Divide $790\,m^{11}\,n^{12}$ by $10\,m^5\,n^7$.
20. Divide $927\,x^{15}\,y$ by $9\,x^{14}\,y$.

SECTION XIII.

DIVISION OF POLYNOMIALS.

Art. **41**. If $a+b-c$ be multiplied by m, the product is $am+bm-cm$; therefore, if $am+bm-cm$ be divided by m, the quotient is $a+b-c$.

A simple quantity is a factor of a polynomial, when it is a factor of every term of that polynomial; and the division of a polynomial by a simple quantity, consists merely in dividing each term of the former by the latter. But a question arises, how we are to determine, in all cases, the signs of the partial quotients. The following considerations will enable us to decide that point.

If $+ab$ be divided by $+a$, the quotient will be $+b$, because $+a$ multiplied by $+b$ gives $+ab$.

If $+ab$ be divided by $-a$, the quotient will be $-b$, because $-a$ multiplied by $-b$ gives $+ab$.

If $-ab$ be divided by $+a$, the quotient will be $-b$, because $+a$ multiplied by $-b$ gives $-ab$.

Lastly, if $-ab$ be divided by $-a$, the quotient will be $+b$, because $-a$ multiplied by $+b$ gives $-ab$.

Hence, the rule for the signs in division, is the same as that in multiplication; that is, *if the two terms, one of which is to be divided by the other, have the same sign, either both + or both —, the quotient must have the sign +; but if they have different signs, that is, one + and the other —, the quotient must have the sign —.*

Art. **42**. Hence, we have the following

RULE FOR DIVIDING A POLYNOMIAL BY A MONOMIAL.

Divide each term of the dividend by the divisor, according to the principles given in Art. 40 for dividing one monomial by another, observing the rule established above for the signs; and the partial quotients taken together, will form the entire quotient.

1. Divide $a^2 b + a m$ by a.
2. Divide $12 x y - 6 x^2 + 3 x y^2$ by $3 x$.
3. Divide $15 a m^4 + 30 m^5 x - 45 m^2$ by $15 m^2$.
4. Divide $36 b^2 c d - 24 b^3 c^2 - 12 b^4 c$ by $6 b^2 c$.
5. Divide $9 a^5 b^6 - 3 a^2 b^3$ by $3 a^2 b^3$.
6. Divide $-21 x^2 y - 7 + 42 x$ by -7.
7. Divide $56 a^4 b^3 c - 28 a^4 b^3 - 168 a^6 b^7 c^2$ by $-28 a^4 b^3$.

Art. **43**. When the dividend and divisor are both polynomials, the process becomes rather more difficult; but, if we observe the manner in which a product is formed by multiplication, we shall readily see the course to be pursued in division.

Multiply $3 a^3 + 2 a^2 b + a b^2$ by $2 a^2 + 3 a b$.

Operation.

$$\begin{array}{lllll}
3a^3 + & 2a^2 b + a b^2 & & & \\
2a^2 + & 3 a b & & & \\
\hline
6a^5 + & 4a^4 b & + 2a^3 b^2 & & \\
 + & 9a^4 b & + 6a^3 b^2 & + 3a^2 b^3 & \\
\hline
6a^5 + & 13 a^4 b & + 8 a^3 b^2 & + 3 a^2 b^3. & \text{Product.}
\end{array}$$

If this product be divided by the multiplicand, the quotient will be the multiplier; or, if it be divided by the multiplier, the quotient will be the multiplicand.

Since each term in the multiplicand is multiplied by each term in the multiplier, if no reduction takes place, the number of terms in the product will be equal to the number produced by multiplying the number of terms in the two factors together. Thus, if one factor have 4 terms and the other 3, the product will contain 12 terms. In most cases, however, a reduction

takes place, by which some terms are united and others wholly disappear.

But there are always two terms, which can neither be united with any others, nor cancelled by any others; viz: one, which is produced by multiplying the term containing the *highest* power of any letter in the multiplicand, by the term containing the *highest* power of the same letter in the multiplier; and the other, which arises from the product of the terms with the *lowest* exponents of the same letter.

Now, since the dividend is to be considered as the product of the divisor and quotient, it is evident, that if the term containing the highest power of a particular letter in the dividend, be divided by the term containing the highest power of the same letter in the divisor, the result will be the term of the quotient containing the highest power of that letter.

Let us reverse the process of multiplication, and divide $6a^5 + 13a^4b + 8a^3b^2 + 3a^2b^3$ by $3a^3 + 2a^2b + ab^2$.

Operation.

Dividend.

$$\begin{array}{l|l}
6a^5 + 13a^4b + 8a^3b^2 + 3a^2b^3 & 3a^3 + 2a^2b + ab^2. \text{ Divisor.} \\
\underline{6a^5 + 4a^4b + 2a^3b^2} & \overline{2a^2 + 3ab.} \text{ Quotient.} \\
\qquad\quad 9a^4b + 6a^3b^2 + 3a^2b^3 & \\
\qquad\quad \underline{9a^4b + 6a^3b^2 + 3a^2b^3} & \\
\qquad\qquad\qquad 0. &
\end{array}$$

According to the preceding remarks, $6a^5$ divided by $3a^3$ must produce the term containing the highest power of a in the quotient; $3a^3$ is contained $2a^2$ times in $6a^5$; then, as the entire dividend is produced by multiplying the whole of the divisor by the whole of the quotient, if we multiply the whole divisor by this first term $2a^2$ of the quotient, a part of the dividend will be produced. The product of the divisor by $2a^2$ is $6a^5 + 4a^4b + 2a^3b^2$, which being subtracted from the dividend, leaves $9a^4b + 6a^3b^2 + 3a^2b^3$.

This remainder is to be considered as a new dividend, and as produced by multiplying the divisor by the remaining part of the quotient. The first term, $9a^4b$, of this new dividend, must

have been produced by multiplying $3a^3$ by the term containing the next highest power of a in the quotient; therefore, if it be divided by $3a^3$, that term of the quotient will be obtained. Dividing $9a^4b$ by $3a^3$, we have $3ab$ for the second term of the quotient; then, multiplying the divisor by $3ab$, the product is $9a^4b + 6a^3b^2 + 3a^2b^3$, which subtracted from the last dividend, leaves no remainder. The entire quotient therefore is $2a^2 + 3ab$.

We perceive, from the preceding example, that we always divide the term containing the highest power of some letter in the dividend, by the term containing the highest power of the same letter in the divisor. It will, therefore, be found convenient to write the quantities in such a manner, that these two terms may stand, the one on the left of the dividend, and the other on the left of the divisor. This will be accomplished by *arranging* the dividend and divisor according to the powers of the same letter, beginning with the highest.

A polynomial is said to be arranged according to the powers of a particular letter, when the terms are so written, that the powers of that letter go on increasing or diminishing from left to right. Thus, in the example just performed, the quantities were arranged according to the diminishing powers of a.

With regard to the signs of the partial quotients, the same rule is applicable, that was given for the division of a polynomial by a monomial.

Art. **44**. From what precedes we deduce the following

RULE FOR THE DIVISION OF ONE POLYNOMIAL BY ANOTHER.

1. *Arrange the dividend and divisor according to the powers of the same letter, beginning with the highest.*

2. *Divide the first term of the dividend by the first term of the divisor, and place the result as the first term of the quotient; recollecting, that if both terms have the same sign, the partial quotient must have the sign $+$, but if they have different signs, the partial quotient must have the sign $-$.*

3. *Multiply the entire divisor by this term of the quotient, sub-*

tract the product from the dividend, and the remainder will form a new dividend.

4. *Divide the first term of the new dividend by the first term of the divisor, and the result will form the second term of the quotient; multiply the entire divisor by this second term of the quotient, and subtract the product from the second dividend. The remainder will form a new dividend, from which another term of the quotient can be found.*

These operations are to be repeated, until all the terms of the original dividend are exhausted.

N. B. The same arrangement must be preserved, in each of the partial dividends, as was made at first in the whole dividend.

When the first term of any remainder cannot be divided by the first term of the divisor, the process must terminate, unless the quotient be continued in a fractional form. When the division terminates, the remainder, if there be one, may be written over the divisor, in the form of a fraction, and annexed to the entire part of the quotient.

Let it be proposed to divide $75\,a^2\,b^4 - 27\,a\,b^5 - 49\,a^3\,b^3 + 20\,a^5\,b - 19\,a^4\,b^2$ by $4\,a^2\,b + 3\,b^3 - 7\,a\,b^2$.

Operation.

$$\begin{array}{l|l}
20a^5b - 19a^4b^2 - 49a^3b^3 + 75a^2b^4 - 27ab^5 & 4a^2b - 7ab^2 + 3b^3. \\
\underline{20a^5b - 35a^4b^2 + 15a^3b^3} & \overline{5a^3 + 4a^2b - 9ab^2.} \\
\quad + 16a^4b^2 - 64a^3b^3 + 75a^2b^4 - 27ab^5 & \\
\quad \underline{+ 16a^4b^2 - 28a^3b^3 + 12a^2b^4} & \\
\qquad\qquad - 36a^3b^3 + 63a^2b^4 - 27ab^5 & \\
\qquad\qquad \underline{- 36a^3b^3 + 63a^2b^4 - 27ab^5} & \\
\qquad\qquad\qquad 0. &
\end{array}$$

After having arranged the two quantities according to the powers of a, and placed the divisor on the right of the dividend, we divide $20\,a^5\,b$ by $4\,a^2\,b$, which gives $+5\,a^3$ for the first term of the quotient; we then multiply the divisor by $5\,a^3$, write the product under the dividend, and subtract it from the dividend. The subtraction of the product is performed by changing its signs, considering it as written after the dividend, and reducing similar terms; thus, the signs being changed, it becomes

$-20\,a^5 b + 35\,a^4 b^2 - 15\,a^3 b^3$; then, by reduction, $+20\,a^5 b$ and $-20\,a^5 b$ cancel each other, $-19\,a^4 b^2$ and $+35\,a^4 b^2$ make $+16\,a^4 b^2$, and $-49\,a^3 b^3$ and $-15\,a^3 b^3$ make $-64\,a^3 b^3$; bringing down the other terms of the given dividend, we have for a remainder $16\,a^4 b^2 - 64\,a^3 b^3 + 75\,a^2 b^4 - 27\,a b^5$, which forms a new dividend.

We now divide $16\,a^4 b^2$ by the first term of the divisor $4\,a^2 b$, and obtain for the second term of the quotient $+4\,a^2 b$; multiplying the divisor by $4\,a^2 b$, and subtracting the product, $+16\,a^4 b^2 - 28\,a^3 b^3 + 12\,a^2 b^4$, from the second dividend, in the same manner as before, we obtain for a remainder $-36\,a^3 b^3 + 63\,a^2 b^4 - 27\,a b^5$, which forms the third dividend.

We now divide $-36\,a^3 b^3$ by $4\,a^2 b$ and obtain $-9\,a b^2$ for the third term of the quotient; multiplying the divisor by $-9\,a b^2$, and subtracting the product, $-36\,a^3 b^3 + 63\,a^2 b^4 - 27\,a b^5$, from the third dividend, we have no remainder. The entire quotient, therefore, is $5\,a^3 + 4\,a^2 b - 9\,a b^2$.

As another example, let it be proposed to divide $a^5 - b^5$ by $a - b$.

Operation.

$$
\begin{array}{llll}
a^5 - b^5 & \left\{ \begin{array}{l} a - b \\ \hline a^4 + a^3 b + a^2 b^2 + a b^3 + b^4. \end{array} \right. & & \\
a^5 - a^4 b & & & \\
\hline
+a^4 b - b^5 & & & \\
+a^4 b - a^3 b^2 & & & \\
\hline
& +a^3 b^2 - b^5 & & \\
& +a^3 b^2 - a^2 b^3 & & \\
\hline
& & +a^2 b^3 - b^5 & \\
& & +a^2 b^3 - a b^4 & \\
\hline
& & & +a b^4 - b^5 \\
& & & +a b^4 - b^5 \\
\hline
& & & 0.
\end{array}
$$

In the last example, several terms are produced in the course of the operation, which are not found in the dividend; these terms disappear, by reduction, when the quotient and divisor are multiplied together.

1. Divide $a^3 + 3\,a^2 m + 3\,a m^2 + m^3$ by $a + m$.

2. Divide $a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$ by $x^2 + 2ax + a^2$.

3. Divide $m^4 - 4m^3x + 6m^2x^2 - 4mx^3 + x^4$ by $m - x$.

4. Divide $6x^4 - 96$ by $3x + 6$.

5. Divide $2b^4 - 9a^2b^2 + 6a^4 + 4a^3b - 3ab^3$ by $2a^2 + 2ab - b^2$.

6. Divide $20a^5 - 41a^4b + 50a^3b^2 - 45a^2b^3 + 25ab^4 - 6b^5$ by $5a^3 - 4a^2b + 5ab^2 - 3b^3$.

7. Divide $4x^4 - 9a^2x^2 + 6a^3x - a^4$ by $2x^2 + 3ax - a^2$.

8. Divide $a^6 + 2a^3z^3 + z^6$ by $a^2 - az + z^2$.

9. Divide $a^6 - 16a^3x^3 + 64x^6$ by $a^2 - 4ax + 4x^2$.

10. Divide $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

11. Divide $10a^4 - 48a^3b + 51a^2b^2 + 4ab^3 - 15b^4$ by $8ab - 2a^2 - 5b^2$.

12. Divide $m^7 - x^7$ by $m - x$.

13. Divide $5a^5b^3c^5 - 22a^4b^3c^6 + 5a^3b^3c^7 + 12a^2b^3c^8 - 7a^2b^2c^8 + 28ab^2c^9$ by $a^3bc^2 - 4abc^3$.

Art. **45**. In the preceding examples, the division could be exactly performed. Let it now be proposed to divide a by $1 - x$.

Operation.

$$
\begin{array}{l|l}
a & 1 - x \\
\cline{2-2}
a - ax & a + ax + ax^2 + ax^3 + \dfrac{ax^4}{1-x}. \\
\cline{1-1}
\quad + ax & \\
\quad ax - ax^2 & \\
\cline{1-1}
\qquad + ax^2 & \\
\qquad ax^2 - ax^3 & \\
\cline{1-1}
\qquad\quad + ax^3 & \\
\qquad\quad ax^3 - ax^4 & \\
\cline{1-1}
\qquad\qquad + ax^4. &
\end{array}
$$

In the example above, it is manifest that the operation would never terminate. This quotient is called an *infinite series*, and we see that any term, except the first, may be formed by multiplying the preceding term by x, remembering to place the divi-

sor under the last remainder, to indicate the continuation of the division. The mode in which any term may be found by means of the preceding term, is called the *law of the series.*

Let us, for a second example, divide 1 by $1+a$.

Operation.

$$
\begin{array}{llllll}
1 & & & & & \left\{ \dfrac{1+a}{1-a+a^2-a^3+a^4-a^5+\dfrac{a^6}{1+a}.} \right. \\
\underline{1+a} & & & & & \\
-a & & & & & \\
\underline{-a-a^2} & & & & & \\
\quad a^2 & & & & & \\
\quad \underline{a^2+a^3} & & & & & \\
\quad\quad -a^3 & & & & & \\
\quad\quad \underline{-a^3-a^4} & & & & & \\
\quad\quad\quad a^4 & & & & & \\
\quad\quad\quad \underline{a^4+a^5} & & & & & \\
\quad\quad\quad\quad -a^5 & & & & & \\
\quad\quad\quad\quad \underline{-a^5-a^6} & & & & & \\
\quad\quad\quad\quad\quad a^6. & & & & &
\end{array}
$$

Here we perceive, that the law of the series is the same as before, except that the 2d, 4th, 6th, &c. terms have the sign —.

Art. **46**. *The difference between similar powers of two quantities, the exponent of the powers being integral and positive, is divisible by the difference of those quantities.*

Thus, $a^m - b^m$ is divisible by $a - b$.

To prove this, it is only necessary to find the first remainder.

Operation.

$$
\begin{array}{ll}
a^m - b^m & \left\{ \dfrac{a-b}{a^{m-1}} \right. \\
\underline{a^m - a^{m-1}\,b} & \\
\text{First remainder,} \quad a^{m-1}\,b - b^m = b\,(a^{m-1} - b^{m-1}). &
\end{array}
$$

Now it is manifest, that if this remainder, $b\,(a^{m-1} - b^{m-1})$, can be exactly divided by $a-b$, the dividend, $a^m - b^m$, is divisible by it also. But since a product is divided by dividing one

of its factors, this remainder, and consequently $a^m - b^m$, is divisible by $a - b$, if the factor $a^{m-1} - b^{m-1}$ is divisible by it.

That is, if $a - b$ divide $a^{m-1} - b^{m-1}$, it will also divide $a^m - b^m$; or, in other words, if the proposition is true for the $m - 1$th power, it must be true for the mth or next higher power.

But we have already seen (Art. **44**), that it is true for the 5th power; therefore, it is true for the 6th; being true for the 6th, it must also be true for the 7th, and so on. Hence, it must be true in all cases, and $a^m - b^m$ is divisible by $a - b$.

In like manner, $(a+b)^m - (c+d)^m$ is divisible by $(a+b) - (c+d)$.

We continue the operation of dividing $a^m - b^m$ by $a - b$, in order that the learner may see the form of the quotient.

$$
\begin{array}{l|l}
a^m - b^m & a - b \\
\cline{2-2}
 & a^{m-1} + a^{m-2}b + a^{m-3}b^2 \ldots\ldots ab^{m-2} + b^{m-1}. \\
a^m - a^{m-1}b & \\
\hline
\quad a^{m-1}b - b^m & \\
\quad a^{m-1}b - a^{m-2}b^2 & \\
\hline
\qquad a^{m-2}b^2 - b^m & \\
\qquad a^{m-2}b^2 - a^{m-3}b^3 & \\
\hline
\qquad\quad a^{m-3}b^3 - b^m. &
\end{array}
$$

Of course, the number of terms in the quotient must be indefinite, until some determinate value is assigned to m. The points . . . . . are used to supply the place of the indefinite number of terms. It will be easy to perceive from what follows, that the last two terms must be such as we have represented them.

It may be proved, however, that, in any case like the preceding, the number of terms in the quotient, will always be equal to the exponent m. This fact may be deduced from an examination of the successive terms of the quotient.

Since the division can be exactly performed, the last term of the quotient must be such, as, being multiplied by b, the last term of the divisor, will produce b^m, the last term of the dividend; that is, it must be b^{m-1}, for b times b^{m-1} is b^m.

But we see, that, in the successive terms of the quotient, the exponent of a goes on diminishing by unity, as many units being

subtracted from m in each case, as mark the number of the term from the first inclusive; and, that the exponent of b in any case, is always 1 less than the number of the term.

Consequently, in the mth term, the exponent of a must be $m - m$ or 0, and the exponent of b must be $m - 1$. The mth term, therefore, is $a^0 b^{m-1}$ or b^{m-1}, (Art. **39**). But we have seen that the last term must be b^{m-1}. Hence the mth term and the last term are the same; in other words, there are m terms in the quotient.

Thus, $\frac{x^8 - y^8}{x - y} = x^7 + x^6 y + x^5 y^2 + x^4 y^3 + x^3 y^4 + x^2 y^5 + x y^6 + y^7$; the quotient containing 8 terms.

It might also be proved, that the *difference* between similar *even* powers of two quantities, and the *sum* of similar *odd* powers, are each divisible by the *sum* of those quantities.

Thus, $\frac{x^n - y^n}{x + y}$ gives an exact quotient of n terms, when n is an even number; and $\frac{x^n + y^n}{x + y}$ gives an exact quotient of n terms, when n is an odd number.

1. Divide $x^9 - y^9$ by $x - y$.
2. Divide $243 a^5 - 1024 b^5$ by $3 a - 4 b$.
3. Divide $x^{10} - y^{10}$ by $x + y$.
4. Divide $1 - m^8$ by $1 + m$.
5. Divide $m^5 + n^5$ by $m + n$.
6. Divide $m^9 + 1$ by $m + 1$.
7. Divide 1 by $1 - m^2$, finding 6 terms of the series, and annexing the remainder placed over the divisor.
8. Divide a by $1 + x y$, finding 6 terms of the series, and annexing the remainder placed over the divisor.

Art. **47**. It is manifest, that when a product is represented in its factors, dividing one of the factors, divides the whole product; also, that we may, in any case, divide the dividend first by one factor of the divisor, then divide the resulting quotient by another, and so on.

Thus, $7 \,.\, 8 \,.\, 3 = 168$; dividing the factor 8 by 4, we have $7 \,.\, 2 \,.\, 3 = 42$, which is $\frac{1}{4}$ of 168.

Also, in dividing $6 \,.\, 12$ or 72 by $2 \,.\, 4$ or 8, we may first divide by 2, which gives $3 \,.\, 12$, and then this quotient by 4, which gives $3 \,.\, 3 = 9 = \frac{72}{8}$.

In like manner, $(m - n)\,(a^2 - b^2)$ divided by $a - b$, gives $(m - n)\,(a + b)$, to obtain which, we divide the factor $a^2 - b^2$ by the divisor.

Also, $(x^2 - y^2)\,(a^2 + 2\,a\,b + b^2)$ divided by $(x + y)\,(a + b)$, gives $(x - y)\,(a + b)$; to obtain which, we divide the factor $x^2 - y^2$ by $x + y$, and the factor $a^2 + 2\,a\,b + b^2$ by $a + b$.

1. Divide $a^2\,(x + y)$ by $x + y$.
2. Divide $(m^2 - 1)\,(a + b)$ by $m - 1$.
3. Divide $14\,(x^3 - 1)\,(a + m)$ by $7\,(x - 1)$.
4. Divide $(a^2 - b^2)\,(x^3 + y^3)$ by $(a - b)\,(x + y)$.
5. Divide $27\,(m^4 - n^4)\,(x^5 + y^5)$ by $3\,(m^2 + n^2)\,(x + y)$.
6. Divide $30\,m^2\,(x^6 - y^6)\,(m^4 - n^4)$
   by $10\,m\,(x^3 + y^3)\,(m^2 - n^2)$.
7. Divide $25\,(a + b)\,(m^4 - 1)\,(x^5 + 1)$
   by $5\,(a + b)\,(m^2 + 1)\,(x + 1)$.

SECTION XIV.

MULTIPLICATION OF FRACTIONS BY INTEGRAL QUANTITIES.

Art. 48. Fractions have the same signification in algebra, that they have in arithmetic. Thus, $\frac{a}{b}$ signifies that one unit is divided into b equal parts, and that a of those parts are used; or, it represents division, and signifies that a is divided into b equal parts.

How much is 5 times $\frac{2}{11}$? Ans. $\frac{10}{11}$.

How much is 3 times $\frac{a}{b}$? Ans. $\frac{3\,a}{b}$.

How much is c times $\frac{a}{b}$? Ans. $\frac{ac}{b}$.

What is $\frac{3}{5}$ of 4? $\frac{1}{5}$ of 4 is $\frac{4}{5}$, and $\frac{3}{5}$ of 4 is $\frac{12}{5}$. Ans.

What is $\frac{2}{7}$ of a? $\frac{1}{7}$ of a is $\frac{a}{7}$, and $\frac{2}{7}$ of a is $\frac{2a}{7}$. Ans.

What is the $\frac{a}{b}$ part of c? $\frac{1}{b}$ of c is $\frac{c}{b}$, and $\frac{a}{b}$ of c is $\frac{ac}{b}$.

In the first three questions, the object was to multiply a fraction by an integral quantity, and in the last three, to find a fractional part of a quantity, that is, to multiply an integral quantity by a fraction; and we perceive that both objects were accomplished by multiplying the numerator and the integral quantity together.

Hence, to multiply a fraction by an integral quantity, or an integral quantity by a fraction; multiply the numerator by the integral quantity, and write the product over the denominator.

1. Multiply $\frac{ax}{m}$ by bc.
2. Multiply $\frac{b+c}{a}$ by mx.
3. Multiply $\frac{a+b}{c}$ by $m+n$.
4. Multiply $\frac{xy}{a-x}$ by $4c+3x$.
5. Multiply $\frac{m+n}{b-c}$ by $m+n$.
6. Multiply $\frac{a}{b^2+2ac}$ by $12a^3+25a^2b$.
7. What is the $\frac{x+y}{b+c}$ part of a^2-b^2?
8. What is the $\frac{2ac-2bc}{m^2+2mn+n^2}$ part of $2ac+2bc$?
9. Multiply $25x^2+13xy$ by $\frac{2ax+5ax^2}{17b+3c}$.

10. Multiply $\frac{a}{15}$ by 5.

The fraction $\frac{a}{15}$ signifies that a is divided into 15 equal parts; but if the denominator be divided by 5, a will then be divided into 3 parts, or $\frac{1}{5}$ as many parts as it was before; consequently, each of the parts will be 5 times as great as before; that is, the fraction $\frac{a}{3}$ is 5 times $\frac{a}{15}$.

11. Multiply $\frac{a}{b\,c}$ by b.

Here a is divided into $b\,c$ equal parts; but if the denominator be divided by b, a will then be divided into $\frac{1}{b}$ as many parts as it was before; the parts, therefore, will be b times as great as they were before; that is, the fraction $\frac{a}{c}$ is b times $\frac{a}{b\,c}$.

Hence, to multiply a fraction and an integral quantity together, divide the denominator by the integral quantity, if possible.

Art. **49**. Combining this rule with the preceding, we have a

GENERAL RULE TO MULTIPLY A FRACTION AND AN INTEGRAL QUANTITY TOGETHER.

Divide the denominator by the integral quantity, if it can be done; if not, multiply the numerator by the integral quantity.

The following examples may be performed by dividing the denominators.

1. Multiply $\frac{4\,a + 3\,x\,y}{m^2\,x}$ by m.

2. Multiply $x^2\,y$ by $\frac{3\,a\,b + 4\,a\,y}{12\,x^3\,y^2 - 19\,x^4\,y^3}$.

3. Multiply $\frac{m^2 - n^2 + a\,b}{a^2 - b^2}$ by $a + b$.

4. Multiply $\frac{14\,a\,c + 25\,b^2 + 43}{21\,a^3\,b^2 + 14\,a^4\,b^3 - 42\,a^2\,b^2\,x\,y}$ by $7\,a^2\,b^2$.

5. Multiply $\frac{4bm^2+11ax^2-93}{x^3-y^3}$ by x^2+xy+y^2.

6. Multiply $\frac{3bc+4xy}{a^4+4a^3b+6a^2b^2+4ab^3+b^4}$ by $a^3+3a^2b+3ab^2+b^3$.

7. Multiply $\frac{3ab}{5(x^4-y^4)}$ by x^2+y^2.

8. Multiply $\frac{7mx+3by}{8(a+b)(a-b)}$ by $4(a+b)$.

9. Multiply $\frac{ax+by}{7(m^6-y^6)}$ by $m+y$.

10. Multiply $\frac{15xy}{26(a+b)(x^5+y^5)}$ by $13(x+y)$.

11. Multiply $\frac{a+bcm}{21(a-b)(c^2-d^2)}$ by $7(a-b)(c-d)$.

12. Multiply $\frac{3ac+xy}{(m^2-n^2)(x^5+y^5)}$ by $(m-n)(x+y)$.

13. Multiply $\frac{a}{b}$ by b.

Dividing the denominator by b, the fraction becomes $\frac{a}{1}$ or a.

14. Multiply $\frac{m^2+bc}{x^2y}$ by x^2y.

Dividing the denominator by x^2y, the fraction becomes $\frac{m^2+bc}{1}$, that is, m^2+bc.

From the last two examples we see, that,

If a fraction is multiplied by a quantity equal to its denominator, the product will be its numerator.

15. Multiply $\frac{a+b}{c}$ by c.

16. Multiply $\frac{m^2+2mc+c^2}{a-b}$ by $a-b$.

17. Multiply $\frac{3a^2bc+4xy}{m^2+3mc}$ by m^2+3mc.

18. Multiply $\frac{4bcm - 12x^2y}{127a}$ by $127a$.

Art. **50**. Multiply $\frac{ac}{m^2x}$ by my.

First multiply by m; the product is $\frac{ac}{mx}$; multiply this product by y, and the result is $\frac{acy}{mx}$ (Art. **49**).

The result would have been the same, if we had divided the integral quantity and the denominator of the fraction both by m, and then proceeded according to the first rule in Art. **48**.

Therefore,

When an integral quantity and a fraction are to be multiplied together, if the integral quantity and denominator of the fraction have common factors, those factors may be omitted in both before multiplying.

1. Multiply $\frac{a^2c}{x^2y^2}$ by mxy.
2. Multiply $\frac{am}{b^2c^3d}$ by ab^2c^3.
3. Multiply ac^2xy by $\frac{x^3y^3}{a^2c^4}$.
4. Multiply $7a^3x^3$ by $\frac{p^2q^5}{x^4y^3}$.
5. Multiply $\frac{a+b}{m^2n}$ by $3m^2x$.
6. Multiply $3my^2(a+b)$ by $\frac{a^4x^3}{6m^3(a+b)}$.
7. Multiply $\frac{bcy}{4(a+b)(x-m)}$ by $4(a+b)(c-d)$.
8. Multiply $\frac{5a+4b}{49(x^2-y^2)}$ by $7ac(x+y)$.
9. Multiply $3(a+m)(a+x)$ by $\frac{a+x}{6(a^2+2am+m^2)}$.
10. Multiply $\frac{bcm^2+ax}{ab(m^4-1)(x-y)}$ by $(a+b)(m^2-1)(x-y)$.

SECTION XV.

DIVISION OF FRACTIONS BY INTEGRAL QUANTITIES.

Art. **51.** Divide $\frac{6}{11}$ by 2; or find $\frac{1}{2}$ of $\frac{6}{11}$. Ans. $\frac{3}{11}$.

Divide $\frac{3c}{b}$ by 3; or find $\frac{1}{3}$ of $\frac{3c}{b}$. Ans. $\frac{c}{b}$.

Divide $\frac{ab}{c}$ by b; or find $\frac{1}{b}$ of $\frac{ab}{c}$. Ans. $\frac{a}{c}$.

For, in each of these examples, if the quotient be multiplied by the divisor, the product will be the dividend.

Hence, to divide a fraction by an integral quantity, divide the numerator by the integral quantity, if possible.

1. Divide $\frac{4a^2c}{7m}$ by $2ac$.
2. Divide $\frac{21x^2y^3}{11bc}$ by $7x^2y$.
3. Divide $\frac{120a^2b^2xy}{4m^2-n^2}$ by $60a^2x$.
4. Divide $\frac{17a^2b^2}{bc+4xy}$ by a^2b^2.
5. Divide $\frac{a^2+ab}{m}$ by a.
6. Divide $\frac{3a^2m+9am^2+12ac}{116+cd}$ by $3a$.
7. Divide $\frac{a^2+2ab+b^2}{x+y}$ by $a+b$.
8. Divide $\frac{a^3-b^3}{3bc-xy}$ by $a-b$.
9. Divide $\frac{12x^3+29x^2+14x}{ab+7b^2}$ by $4x+7$.
10. Divide $\frac{18x^3-33x^2+44x-35}{11m+4n^2}$ by $3x^2-2x+5$.
11. Divide $\frac{a}{b}$ by c.

In the last example, we cannot divide the numerator; but in Art. **48**, we showed that a fraction is multiplied by dividing its denominator; on the other hand, a fraction may be divided by multiplying its denominator.

The fraction $\frac{a}{b}$ signifies that a is divided into b equal parts, and if the denominator be multiplied by 3, for example, a would then be divided into 3 times as many parts, and the parts would be only $\frac{1}{3}$ as great as before; that is, $\frac{a}{3b}$ is $\frac{1}{3}$ of $\frac{a}{b}$. In like manner, if the denominator be multiplied by c, a will be divided into c times as many parts, and the parts will be only $\frac{1}{c}$ as great as before; that is, $\frac{a}{bc}$ is $\frac{1}{c}$ of $\frac{a}{b}$.

Hence, to divide a fraction by an integral quantity, multiply the denominator by the integral quantity.

Art. **52**. Combining this rule with the preceding, we have the following

GENERAL RULE FOR DIVIDING A FRACTION BY AN INTEGRAL QUANTITY.

Divide the numerator, if it can be done; if not, multiply the denominator, by the integral quantity.

1. Divide $\frac{3a}{bc}$ by m.
2. Divide $\frac{x+y}{b-c}$ by $b+c$.
3. Divide $\frac{m+n}{x+y}$ by $4x^2$.
4. Divide $\frac{3ab+4b}{3x-4m}$ by $3x+4m$.
5. Divide $\frac{x^2+2xy+y^2}{m+n}$ by $x+y$.
6. Divide $\frac{x^2-y^2}{a+b}$ by $x-y$.

7 Divide $\dfrac{3+7x}{4+bc}$ by $p+q$.

8. Divide $\dfrac{a^2-b^2}{x^2+xy+y^2}$ by $x-y$.

9. Divide $\dfrac{14m^2+21x}{b^2+c^2}$ by 7.

10. Divide $\dfrac{x^3-y^3}{a^2}$ by $x-y$.

11. Divide $\dfrac{m^4-n^4}{x+y}$ by $m+n$.

Art. **53**. Divide $\dfrac{10a^2m}{7xy}$ by $6mn$.

The divisor is the same as $2.3mn$. First divide by $2m$; the quotient is $\dfrac{5a^2}{7xy}$. Divide this quotient by $3n$, the remaining factors of the divisor; the result is $\dfrac{5a^2}{21nxy}$.

The result would have been the same, if we had first divided the numerator of the fraction and the integral quantity both by $2m$, their common factors, and then proceeded according to the second rule in Art. **51**.

Hence, when a fraction is to be divided by an integral quantity, if the numerator of the fraction and the integral quantity have common factors, those factors may be omitted in both, previous to the division.

1. Divide $\dfrac{2a^2bx}{m^4}$ by $3bx$.

2. Divide $\dfrac{15x^2y^2}{7}$ by $5abx$.

3. Divide $\dfrac{27m^3x^2y^3}{5a^2b^2}$ by $9abmxy^2$.

4. Divide $\dfrac{36x^2}{7y^2}$ by $36ax^2$.

5. Divide $\dfrac{14m^3n}{x+y}$ by $2mx$.

6. Divide $\frac{m^2(a+b)}{x+y}$ by $4m(a+b)$.

7. Divide $\frac{x^2-y^2}{3m}$ by $7n(x+y)$.

8. Divide $\frac{4(x^2-y^2)(a+b)}{3m^2}$ by $2a(x+y)$.

9. Divide $\frac{6(a+b)(m^3-n^3)}{5a}$ by $3xy(a+b)(m-n)$.

SECTION XVI.

FACTORS AND DIVISORS OF ALGEBRAIC QUANTITIES.

Art. **54.** A *prime quantity* in algebra, like a prime number in arithmetic, is one which is divisible by no entire and rational quantity, except itself and unity. Thus, a, b and $a+b$ are prime quantities; but ab is not prime, because it is divisible both by a and b.

Two quantities are said to be *prime with regard to each other*, when the same quantity will not exactly divide them both, that is, without a remainder. Thus, ab and cm, although neither of them is a prime quantity, are prime with respect to each other.

It is to be observed, however, that when we call a, b, c, &c., prime quantities, we mean simply that they cannot be algebraically divided by other quantities, except by representation; but they are, strictly speaking, prime quantities, only when they represent *prime numbers*.

Sometimes it is requisite to separate quantities into their prime factors. This operation, in monomials, is attended with no difficulty; for we have only to ascertain, according to the method usually given in arithmetic, the prime factors of the coëfficient, and to represent them as multiplied together and followed by the several literal quantities, each written as many times as it is a factor.

For example, $12a^2b^3 = 3.2^2a^2b^3 = 3.2.2aabbb$. In

this quantity the different prime factors are 3, 2, a and b; 3 is contained once, 2 twice, a twice and b three times as a factor.

The prime factors would, however, be sufficiently indicated, merely by ascertaining those of the coëfficient; for the exponents show how many times the letters are contained as factors. Thus, $54\,a^2 b^5 c^4 = 2\,.\,3\,.\,3\,.\,3\,a^2 b^5 c^4 = 2\,.\,3^3\,a^2 b^5 c^4$.

When a quantity is the product of a monomial and a prime polynomial, in order to separate it into factors, it is only necessary to divide it by the greatest monomial, that will exactly divide all the terms, and to place the divisor, separated into prime factors, before the quotient, the latter being included in a parenthesis.

Thus, $a b + b^2 = b\,(a + b)$; in which the factors are b and $a + b$. In like manner, $4\,b\,c^2 + 8\,b^2\,c = 4\,b\,c\,(c + 2\,b) = 2^2\,b\,c\,(c + 2\,b)$; in this case, the prime factors are, 2, b, c and $c + 2\,b$.

Let the learner separate the following quantities into prime factors.

1. $a\,m - b\,m$. Ans. $m\,(a - b)$.
2. $a\,b + a\,c - 2\,a\,m$.
3. $4\,a\,x + 2\,x\,y + 12\,x$.
4. $25\,x^2 + 30\,x^2\,y - 15\,x^2\,m$.
5. $81\,a\,b\,c + 27\,a^2\,b^2\,c + 54\,a^3\,b\,c$.
6. $99\,m^2\,x\,y + 108\,m^2\,p\,q + 18\,m^2\,r$.
7. $12\,a\,b + 24\,a\,b\,c - 36\,a\,b\,x$.

Art. **55**. When a quantity is the product of two or more polynomials, the process of finding its factors becomes more difficult; but there are cases in which some of the factors, at least, may be easily ascertained.

1. Any quantity which is known to be a *power* of a polynomial, may be separated into as many factors, each equal to that polynomial, as there are units in the exponent of the power. Thus, $a^2 + 2\,a\,b + b^2 = (a + b)^2 = (a + b)\,(a + b)$; also, $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 = (a + b)^3 = (a + b)\,(a + b)\,(a + b)$.

2. The *difference* between the second powers of two quantities

may be separated into two factors, one of which is the *sum* and the other the *difference* of the two quantities. For example, $a^2 - b^2 = (a + b)(a - b)$, and $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$. Also, $25\,a^2 x^4 - 36\,b^4 c^8 = (5\,a\,x^2 + 6\,b^2 c^4)(5\,a\,x^2 - 6\,b^2 c^4)$.

3. The *difference* between *similar* powers of two quantities, can be separated into two factors, one of which is the difference of those quantities (Art. **46**). Thus, $a^2 - b^2$, $a^3 - b^3$, $a^4 - b^4$, $a^5 - b^5$, &c. are all divisible by $a - b$.

4. The *difference* between similar *even* powers of two quantities, the powers being above the second, can always be separated into at least three factors, one of which is the *sum* and another the *difference* of the two quantities (Art. **33**). Thus, $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$.

5. The *sum* of similar *odd* powers of two quantities, may be separated into two factors, one of which is the *sum* of the quantities (Art. **46**).

Thus, $x^5 + y^5 = (x + y)(x^4 - x^3 y + x^2 y^2 - x y^3 + y^4)$.

The quantity $x^6 - y^6$ can be separated into four factors. For, $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$. But, $x^3 + y^3 = (x^2 - x y + y^2) \times (x + y)$; and, $x^3 - y^3 = (x^2 + x y + y^2)(x - y)$. Hence, $x^6 - y^6 = (x^2 - x y + y^2)(x + y)(x^2 + x y + y^2)(x - y)$.

We have shown how to find the prime factors of simple quantities, and, in some cases, those of polynomials; but any quantity which will exactly divide another, is a factor of this last, so that some quantities may be separated into factors, not prime, in several different ways. For example, $a^2 b\,c + a\,b^2 c^2 = a\,(a\,b\,c + b^2 c^2) = a\,b\,(a\,c + b\,c^2) = b\,(a^2 c + a\,b\,c^2) = b\,c\,(a^2 + a\,b\,c) = a\,b\,c\,(a + b\,c)$.

Art. **56**. Sometimes it is desirable to find *all* the divisors of a number or of an algebraic quantity. To explain the process by which this may be effected, let us take the quantity $a^2 b^3$. If we include unity and the quantity itself among the divisors, it is evident that $a^2 b^3$ is divisible not only by 1, a and a^2, but also by all the combinations of these with the different powers of b, from

the 1st to the 3d inclusive; that is, each of the factors, 1, a, and a^2, is to be taken once, b times, b^2 times, and b^3 times. Hence, if $1+a+a^2$ be multiplied by $1+b+b^2+b^3$, which is expressed thus, $(1+a+a^2)(1+b+b^2+b^3)$, the different terms of the product will constitute all the divisors of a^2b^3.

Moreover, as no two terms of this product can be similar, the number of them will be equal to the product of the number of terms in one factor by the number of terms in the other. There being 3 terms in one factor and 4 in the other, the product will contain $3 \,.\, 4$ or 12 terms. Performing the multiplication, we find the following quantities for divisors, viz: $1, a, a^2, b, ab, a^2b, b^2, ab^2, a^2b^2, b^3, ab^3, a^2b^3$.

If however any one of the prime factors is a polynomial, all its terms are to be considered as a single quantity in forming the divisors.

Art. **57**. We see, therefore, *that the number of divisors of any quantity may be found, by adding* 1 *to the exponent of each of its prime factors, and multiplying these sums together; also, that the divisors themselves will be the different terms of the product*, $(1+\mathrm{a}+\mathrm{a}^2+\ldots\ldots+\mathrm{a}^{\mathrm{n}})(1+\mathrm{b}+\mathrm{b}^2+\ldots\ldots+\mathrm{b}^{\mathrm{n}'}) \times (1+\mathrm{c}+\mathrm{c}^2+\ldots+\mathrm{c}^{\mathrm{n}''})$ *&c.,** a, b, c *&c. being the prime factors of the quantity in question, and* n, n′, n″ *&c. their exponents.*

For example, let the divisors of $4m^3x^2y$ or $2^2m^3x^2y$ be required. The number of divisors is equal to $(2+1)(3+1) \times (2+1)(1+1) = 3 \,.\, 4 \,.\, 3 \,.\, 2 = 72$; and the divisors themselves will be the terms, obtained by developing the product, $(1+2+4)(1+m+m^2+m^3)(1+x+x^2)(1+y)$.

Let the divisors of $3a^2b+6a^2c$ be required. Now, $3a^2b+6a^2c = 3a^2(b+2c)$; hence, the number of divisors is equal to $2 \,.\, 3 \,.\, 2 = 12$. The divisors will be the terms of $(1+3) \times (1+a+a^2)(1+\overline{b+2c})$; or $1, 3, a, 3a, a^2, 3a^2, (b+2c), 3(b+2c), a(b+2c), 3a(b+2c), a^2(b+2c), 3a^2(b+2c)$.

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* It is to be observed that n' is read n prime, n'' is read n second, &c. The accents, as these marks are called, are used merely to enable us to represent different quantities by the same letter.

As an example in numbers, let the divisors of 160 be required. Separating it into prime factors, $160 = 2 \,.\, 2 \,.\, 2 \,.\, 2 \,.\, 2 \,.\, 5 = 2^5 \,.\, 5$; therefore the number of divisors is $6 \,.\, 2 = 12$; and the divisors are the terms of $(1 + 2 + 4 + 8 + 16 + 32)(1 + 5)$, or 1, 2, 4, 8, 16, 32, 5, 10, 20, 40, 80, 160.

Let the learner find the divisors of the following quantities.

1. $a^2 b^2$.
2. $a^2 b^2 c^3$.
3. $9 x^2 y^3$.
4. $50 m^2$.
5. 150.
6. 780.
7. $4 a^2 b + 16 a b$.
8. $10 a b + 25 a m$.

SECTION XVII.

GREATEST COMMON DIVISOR.

Art. **58**. Any quantity which will exactly divide two or more quantities, is called a *common divisor* of these quantities; and the greatest that will so divide them, is called their *greatest common divisor*.

Suppose A and B two quantities, of which we wish to find the greatest common divisor, A being greater than B.

The learner must remember that A and B are merely concise representations of any two quantities, such as are given in either of the examples succeeding this explanation.

First divide A by B, and if it gives an exact quotient, B itself must be the greatest common divisor; for no quantity can have a divisor greater than itself. But if B will not divide A exactly, suppose that we obtain a quotient Q and a remainder R. Then,

$$A = B\,Q + R.$$

For the dividend must be equal to the product of the divisor and quotient, plus the remainder. By transposition,

$$R = A - B\,Q.$$

Now any quantity which will divide B, must divide Q times B; hence, any quantity which will divide A and B, must exact-

ly divide $A - B\,Q$, and consequently it must divide R exactly, since $R = A - B\,Q$. We see, then, that B and R will have the same common divisor as A and B.

Dividing now B by R, if there is a remainder R', it may be shown as before, that R and R' will have the same common divisor as B and R, that is, the same as A and B.

If we continue to divide the preceding divisor by the remainder, we shall, provided the given quantities have a common divisor, eventually obtain a remainder, which will exactly divide the preceding divisor, and which, consequently, must be the greatest common divisor of itself and that divisor; that is, the greatest common divisor of A and B.

Art. **59**. We have therefore the following

RULE FOR FINDING THE GREATEST COMMON DIVISOR OF TWO QUANTITIES.

Divide the greater by the less, and if there is no remainder, the less quantity will be the divisor sought; but if there is a remainder, divide the first divisor by it, and continue thus to make the preceding divisor the dividend, and the remainder the divisor, until a remainder is obtained, which will exactly divide the preceding divisor; this last remainder will be the greatest common divisor required.

If there are several quantities whose greatest common divisor is required; first find the greatest common divisor of two of them; then find the greatest common divisor of this divisor and one of the other quantities, and so on; the last divisor thus found will be the greatest common divisor sought.

The greatest common divisor of monomials, as well as that of some polynomials, may frequently be found by inspection.

The application of the preceding rule to polynomials, sometimes leads to complicated operations, unsuited to an elementary treatise. We shall, however, give a few examples.

It is to be remarked, that, if one of the quantities under consideration, have a factor not found in the other, this factor may be left out without affecting the common divisor; because this divisor can have no factors, not common to both the quantities.

Moreover, one of the quantities may be multiplied by any quantity, which is not a factor of the other, and which does not contain a factor of the other. This is often necessary, in order to render the division possible. Thus a, which is the greatest common divisor of $a\,m$ and $a\,d$, is also that of $a\,m\,c$ and $a\,d$.

Let it be required to find the greatest common divisor of $6\,a^3 - 6\,a^2\,y + 2\,a\,y^2 - 2\,y^3$ and $12\,a^2 - 15\,a\,y + 3\,y^2$.

We see that 2 is a factor of the first, but not of the second; and, that 3 is a factor of the second, but not of the first. Leaving out these factors, we proceed to divide $3\,a^3 - 3\,a^2\,y + a\,y^2 - y^3$ by $4\,a^2 - 5\,a\,y + y^2$.

Operation.

$3\,a^3 - 3\,a^2\,y + a\,y^2 - y^3$. Multiply by 4 to render the 1st term
4 [divisible by $4\,a^2$.

$$12\,a^3 - 12\,a^2\,y + 4\,a\,y^2 - 4\,y^3 \left\{ \frac{4\,a^2 - 5\,a\,y + y^2}{3\,a + 3} \right.$$

$\underline{12\,a^3 - 15\,a^2\,y + 3\,a\,y^2}$

$3\,a^2\,y + a\,y^2 - 4\,y^3$. Divide by y, because it is not a [factor of the divisor.

$3\,a^2 + a\,y - 4\,y^2$. Multiply by 4 to render division
4 [possible.

$12\,a^2 + 4\,a\,y - 16\,y^2$

$\underline{12\,a^2 - 15\,a\,y + 3\,y^2}$

$19\,a\,y - 19\,y^2$. Divide by $19\,y$.

$a - y$.

We now make $a - y$ the new divisor and $4\,a^2 - 5\,a\,y + y^2$ the dividend.

$$4\,a^2 - 5\,a\,y + y^2 \left\{ \frac{a - y}{4\,a - y} \right.$$

$\underline{4\,a^2 - 4\,a\,y}$

$-a\,y + y^2$

$\underline{-a\,y + y^2}$

$0.$

Hence $a - y$ is the greatest common divisor sought.

Find the greatest common divisors in the following examples.

1. $4a^2bc^2$ and $12a^3b^2c$.
2. $75abc^2$ and $35a^3xy$.
3. $210a^3$, $375axy$ and $45ax^2$.
4. m^4-1 and m^6+m^4.
5. x^2-1 and x^3+1.
6. $4x^4-16x^2$ and $9x+18$.
7. $2x^3-16x-6$ and $3x^3-24x-9$.
8. $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$.
9. $a^3-a^2b+3ab^2-3b^3$ and $a^2-5ab+4b^2$.

SECTION XVIII.

LEAST COMMON MULTIPLE.

Art. **60.** When one quantity can be exactly divided by another, the former is called a multiple of the latter. A common multiple of two or more quantities, is one which is divisible by them all; and the least common multiple is the least quantity divisible by them all.

Suppose that it is required to find the least common multiple of $4a^2b^3$ and $6a^4b$. It is evident that the quantity sought must contain all the factors of each of the given quantities. Separating these into prime factors, $4a^2b^3=2.2a^2b^3=2^2a^2b^3$, and $6a^4b=2.3a^4b$. The different prime factors are 2, 3, a and b, and the multiple required must contain as a factor each of these, as many times as it is found in either of the given quantities; that is, it must contain 2 twice, 3 once, a four times, and b three times as a factor. Consequently, $2.2.3a^4b^3$ or $12a^4b^3$ will be the least common multiple required, for it is manifestly the least quantity divisible by $4a^2b^3$ and $6a^4b$.

Art. **61.** Hence we deduce the following

RULE FOR FINDING THE LEAST COMMON MULTIPLE OF SEVERAL QUANTITIES.

First separate the quantities into their prime factors; then

collect into one product all these different factors, each raised to the highest power found in either of the quantities.

Required the least common multiple in each of the following examples.

1. $a^2 b c$, $2 a^2 b^2 c$, $4 a^3 c$. Ans. $2 . 2 a^3 b^2 c, = 4 a^3 b^2 c$.
2. $9 m^4 y$, $6 a^2 m y^2$, $12 a^3 m^2 y^3$.
3. $25 a^2 m x^3$, $15 a m^3 x$, $30 a^3 m$.
4. 18, $6 x^2 y^3$, $12 x y^5$, $9 a x^2$.
5. $4 a b + 2 a c$, $12 a b$.
6. $9 m^2 + 18 m^3 x$, $27 m x^4$.
7. $a^2 - b^2$, $a^2 + 2 a b + b^2$.
8. $a^3 - b^3$, $a^2 - b^2$.

Remark. It might be proved that the least common multiple of two quantities, is equal to their product divided by their greatest common divisor.

SECTION XIX.

REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

Art. **62**. If both numerator and denominator of a fraction be multiplied by the same quantity, the value of the fraction will not be changed; for, multiplying the numerator multiplies the fraction, and multiplying the denominator divides the fraction; but if a quantity be multiplied, and the product be divided by the multiplier, the value will remain the same as at first.

Also, if the numerator and denominator of a fraction be divided by the same quantity, the value of the fraction will not be changed; for, dividing the numerator divides the fraction, and dividing the denominator multiplies the fraction; but if a quantity be divided, and the quotient be multiplied by the divisor, the value will remain the same as at first.

Art. **63**. From the principle last stated, we derive the following

RULE FOR REDUCING A FRACTION TO ITS LOWEST TERMS.

Divide both numerator and denominator by their greatest common divisor.

Reduce the following fractions to their lowest terms.

1. $\frac{3\,a\,b}{12\,a\,c}$. Ans. $\frac{b}{4\,c}$.

2. $\frac{9\,x^2\,y^3}{15\,x\,y^4}$.

3. $\frac{57\,a^4\,b\,c}{38\,a\,b\,m^2}$.

4. $\frac{155\,x^2\,m^4\,y^2}{75\,x^3\,m\,y^4}$.

5. $\frac{144\,m^2\,x\,y}{1728\,m^3\,x^2\,y^3}$.

6. $\frac{33\,a\,m\,x}{231\,a^4\,m^2\,x}$.

7. $\frac{a\,m\,x + 35\,x^2}{b\,c\,x^2}$.

8. $\frac{7\,x^2\,y^2 + 14\,x\,y}{21\,x^3\,y^3}$.

9. $\frac{125\,a^5\,b\,c^2}{15\,a^3 - 25\,a^4\,b\,c}$.

10. $\frac{2\,a\,b + 13\,a\,m^2}{144\,a^2}$.

11. $\frac{3\,a^2\,x}{9\,a^3\,x^2 - 6\,a^4\,x}$.

12. $\frac{99\,a^2\,m^2}{33\,a^3 - 66\,a\,m^2}$.

In the preceding examples the greatest common divisors of the numerators and denominators are monomials; in those which follow, the greatest common divisors are polynomials, but the quantities may be easily separated into factors, so as to exhibit the common divisors (Art. **55**).

13. $\frac{4\,a^2 - 4\,b^2}{3\,a - 3\,b}$. This is the same as $\frac{4\,(a^2 - b^2)}{3\,(a - b)}$

$$\frac{4\,(a-b)\,(a+b)}{3\,(a-b)} = \frac{4\,(a+b)}{3} = \frac{4\,a + 4\,b}{3}.$$

14. $\frac{3\,x^2\,y + 3\,x\,y^2}{3\,x^2 + 6\,x\,y + 3\,y^2}$.

15. $\frac{a^2 - b^2}{a^2 - 2\,a\,b + b^2}$.

16. $\frac{5\,a^3 + 10\,a^2\,b + 5\,a\,b^2}{8\,a^3 + 8\,a^2\,b}$.

17. $\frac{a^3 - x^3}{a^2 - 2\,a\,x + x^2}$.

18. $\frac{12\,(x^2 - y^2)}{16\,(x^3 + y^3)}$.

19. $\frac{45\,a^3\,b^3\,(x + y)}{25\,a\,b\,(x^4 - y^4)}$.

Art. **64**. The actual division of one monomial by another is impossible, when the coëfficient of the former is not divisible by that of the latter, when the divisor contains any letter not found

in the dividend, or when the exponent of any letter in the divisor exceeds the exponent of the same letter in the dividend. A polynomial cannot be divided by a monomial, unless the latter will divide every term of the former; and the complete division of one entire polynomial by another is impossible, whenever the first term of the original or of any partial dividend, cannot be divided by the first term of the divisor, the quantities being arranged according to the powers of the same letter.

In such cases the division is expressed in the form of a fraction, the divisor being placed under the dividend. The result should then be reduced to its lowest terms.

With regard to polynomials, however, we sometimes partially execute the division, placing the last remainder over the divisor, and annexing it to the entire part of the quotient.

Express the division in the following examples and reduce the results.

1. Divide $3a^2b$ by $7ab$. Ans. $\frac{3a}{7}$.
2. Divide $13xy$ by $26abc$.
3. Divide $33xy^2$ by $66x^2y^3$.
4. Divide $45abc$ by $180abx^2$.
5. Divide abc by $m+n$.
6. Divide $46x^2$ by $4x^2y+6xy^2$.
7. Divide $3x+3y$ by $6(x^2-y^2)$.
8. Divide $12x^2+12y^2$ by $36x^2+36y^2$.
9. Divide $13(x-y)$ by $39(x^3-y^3)$.

SECTION XX.

MULTIPLICATION OF FRACTIONS BY FRACTIONS.

Art. **65**. What is $\frac{2}{3}$ of $\frac{7}{9}$? $\frac{1}{3}$ of $\frac{7}{9}=\frac{7}{27}$, and $\frac{2}{3}$ of $\frac{7}{9}=\frac{14}{27}$.

What is the $\frac{a}{b}$ part of $\frac{c}{d}$? The $\frac{1}{b}$ part of $\frac{c}{d}=\frac{c}{bd}$, and the

$\frac{a}{b}$ part of $\frac{c}{d}=\frac{a\,c}{b\,d}$; that is, $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\,c}{b\,d}$. In like manner, $\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{m}{n}=\frac{a\,c\,m}{b\,d\,n}$.

Hence we deduce the

RULE FOR MULTIPLYING FRACTIONS BY FRACTIONS.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

Remark. As the results should be reduced to the lowest terms, it is convenient to represent the operation and omit the common factors, previous to the actual performance of the multiplication. Thus, $\frac{3\,a\,b}{a+b}\cdot\frac{a+b}{6\,b\,m}=\frac{3\,a\,b\,(a+b)}{6\,b\,m\,(a+b)}=\frac{a}{2\,m}$. Also, $\frac{3\,a}{7}\cdot\frac{14}{5\,b\,m}\cdot\frac{8\,b}{9\,a^3}=\frac{3\,.\,14\,.\,8\,a\,b}{7\,.\,5\,.\,9\,a^3\,b\,m}=\frac{1\,.\,2\,.\,8}{1\,.\,5\,.\,3\,a^2\,m}=\frac{16}{15a^2m}$.

Another mode of proceeding is the following, viz: when fractions are to be multiplied together, if the numerator of one and the denominator of another have common factors, omit those factors before multiplying.

Thus, if $\frac{36\,a^2\,b}{5\,x^2\,y}$ is to be multiplied by $\frac{25\,x\,y^3}{16\,a\,b^3\,c}$, divide the numerator of the first and the denominator of the second by $4\,a\,b$; also, the numerator of the second and the denominator of the first by $5\,x\,y$. The fractions then become $\frac{9\,a}{x}$ and $\frac{5\,y^2}{4\,b^2\,c}$, the product of which is $\frac{45\,a\,y^2}{4\,b^2\,c\,x}$.

1. Multiply $\frac{2\,b\,c}{9}$ by $\frac{6\,a\,b^3}{7\,c^2}$.
2. Multiply $\frac{4\,x^2\,y}{7\,m^2}$ by $\frac{12\,x\,y^2}{5\,b\,c}$.
3. Multiply $\frac{4\,x+3\,b^2}{8\,x}$ by $\frac{24\,x^3}{76+9\,m^2}$.
4. Multiply $\frac{4\,x+1}{3}$ by $\frac{6\,x}{7}$.

5. Multiply $\frac{a^2-b^2}{56}$ by $\frac{3a^2}{a+b}$.

6. Multiply $\frac{3x^2-4x}{14a^2}$ by $\frac{7a}{2x^3-3x}$.

7. Multiply $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$.

8. Find the product of $\frac{2b}{c^2}$, $\frac{7c^3}{6b^2}$ and $\frac{5x^2y}{63m^4}$.

9. Find the product of $\frac{4+x}{y^3}$, $\frac{3y^2}{a+b}$ and $\frac{a+b}{4+x}$.

10. Find the product of $\frac{39x^2}{7a^2}$, $\frac{a^2-x^2}{13x}$ and $\frac{49abc}{a+x}$.

11. Find the product of $\frac{64abcx}{9m^3}$, $\frac{a^2+2ax+x^2}{16a^2}$ and $\frac{3m}{a^2-x^2}$.

12. Find the product of $\frac{4(x^3+y^3)}{x-y}$, $\frac{x^2-y^2}{m^2}$ and $\frac{3a^2m^3}{8(x+y)}$.

SECTION XXI.

ADDITION AND SUBTRACTION OF FRACTIONS. COMMON DENOMINATOR.

Art. **66**. To represent the addition and subtraction of fractions, we merely write them after each other, with the signs $+$ and $-$ between them, being careful to place these signs even with the line separating the numerator and denominator. Thus, $\frac{a}{b}+\frac{c}{d}-\frac{e}{f}$. But if the denominators are alike, the addition and subtraction may be performed upon the numerators.

Add together $\frac{2}{11}$ and $\frac{4}{11}$. Ans. $\frac{2+4}{11}=\frac{6}{11}$.

Subtract $\frac{4}{13}$ from $\frac{7}{13}$. Ans. $\frac{7-4}{13}=\frac{3}{13}$.

Add together $\frac{a}{c}$ and $\frac{b}{c}$. Ans. $\frac{a+b}{c}$.

Subtract $\frac{b}{m}$ from $\frac{c}{m}$. Ans. $\frac{c-b}{m}$.

Add together $\frac{a}{b}$ and $\frac{c}{d}$. Here the denominators are different; but if the numerator and denominator of the first fraction be multiplied by d, and the numerator and denominator of the second be multiplied by b, the denominators will be made alike, without changing the value of the fractions. The first fraction becomes $\frac{ad}{bd}$ and the second $\frac{bc}{bd}$; then adding, we have $\frac{ad+bc}{bd}$. Ans.

Add together $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$. If we multiply the numerator and denominator of each fraction by the denominators of both the others, the fractions become $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$ and $\frac{bde}{bdf}$, the sum of which is $\frac{adf+bcf+bde}{bdf}$, Ans. Hence we derive a

RULE FOR THE ADDITION AND SUBTRACTION OF FRACTIONS.

Reduce them to a common denominator, and then add or subtract the numerators.

Art. **67**. The preceding examples give also the following rule for reducing fractions to a common denominator.

Multiply the denominators together for a common denominator, and multiply each numerator by all the denominators except its own. For this is equivalent to multiplying the numerator and denominator of each fraction by the denominators of all the others, which does not alter the value of the fractions (Art. **62**).

This rule for reducing fractions to a common denominator, will uniformly give correct, but not always the simplest results.

Suppose it required to reduce $\frac{a}{4m^3}$, $\frac{b}{6m}$ and $\frac{c}{3m^2x}$ to a com-

mon denominator. The product of the denominators will, in this case, give a common denominator much greater than is necessary. In order to obtain the least, we must, as in arithmetic, find the least common multiple of all the denominators (Art. **61**). The least common multiple of $4m^3$, $6m$ and $3m^2x$ is $2^2 . 3m^3x$; this therefore is the least common denominator sought. To produce this denominator, the first denominator must be multiplied by $3x$, the second by $2m^2x$, and the third by $4m$; these, therefore, are the quantities by which the numerators are respectively to be multiplied. The fractions then become $\frac{3ax}{12m^3x}$, $\frac{2bm^2x}{12m^3x}$ and $\frac{4cm}{12m^3x}$.

Art. **68**. Hence we deduce a

RULE TO REDUCE FRACTIONS TO THE LEAST COMMON DENOMINATOR.

Find the least common multiple of all the given denominators, and this will be the least common denominator sought; then multiply each numerator by the quantity, by which it would be necessary to multiply its denominator, in order to produce the least common denominator.

Remark. The quantity by which any numerator must be multiplied, may be found by dividing the common denominator by the denominator of the given fraction. It is to be observed also, that fractions must be reduced to their lowest terms, before we apply the rule for reducing them to the least common denominator.

1. Add $\frac{p}{q}$, $\frac{m}{n}$ and $\frac{g}{h}$.

2. Add $\frac{3a}{7}$, $\frac{2a}{5}$ and $\frac{x}{y}$.

3. Add $\frac{4}{7ab}$, $\frac{3x}{14a^2}$ and $\frac{11y}{21b^3}$.

4. Add $\frac{9x^2y}{4p^3}$, $\frac{3ab}{6pq^2}$ and $\frac{7xy^2}{42p^2q^2}$.

5. Add $\frac{a+b}{45x^2}$, $\frac{11}{75xy}$ and $\frac{13a}{25x^3y^2}$.

6. Add $\frac{a-b}{27}$, $\frac{a+b}{81}$ and $\frac{a}{18}$.

7. Add $\frac{3xy}{25m^2}$, $\frac{2bc}{15m^3n}$ and $\frac{4hx}{35m^3n^4}$.

8. Add $\frac{3m}{7x}$, $\frac{bc}{5x^2}$, $\frac{c^2}{10x^3}$ and $\frac{4a}{25xy^2}$.

9. Add $\frac{3a^2}{26}$, $\frac{2a}{5}$ and $\frac{3b}{7a}$.

10. Add $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$.

11. Add $\frac{1}{1+x}$ and $\frac{1}{1-x}$.

12. Add $\frac{3}{a^2-b^2}$ and $\frac{7}{a-b}$.

13. Add $\frac{3a}{4b^2}$ and $\frac{5m}{4ab+8a}$.

14. Subtract $\frac{3a}{4m}$ from $\frac{7b}{6m^2}$.

15. Subtract $\frac{a^2-b^2}{49}$ from $\frac{a^2+b^2}{28}$.

16. Subtract $\frac{ax}{b+c}$ from $\frac{ax}{b-c}$.

17. Subtract $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$.

18. Subtract $\frac{4a^2+2b}{56}$ from $\frac{5a^2+b}{36}$.

19. Subtract $\frac{3x-7}{56a}$ from $\frac{4x}{7a^2}$.

20. Subtract $\frac{2ax}{7a-14x}$ from $\frac{4ax}{21}$.

21. Subtract $\frac{1-x^2}{1+x^2}$ from $\frac{1+x^2}{1-x^2}$.

22. Add $\dfrac{bc}{x+y}$ and mp.

In this example, the integral quantity must be reduced to a fraction having $x+y$ for its denominator.

23. Add $a+b$ and $\dfrac{a-b}{4}$.

24. Add m^2+n^2 and $\dfrac{4m^3+3n^3}{7m-2n}$.

25. Add 1 and $\dfrac{3a^2-3b^2}{a^2+b^2}$.

26. Subtract $\dfrac{3a-7b}{4xy}$ from $3m^2$.

27. Subtract $\dfrac{4h^3n^3}{x+1}$ from $5h^3n^3$.

28. Subtract 2 from $\dfrac{7ab+4c^2}{m^3}$.

29. Subtract $a+b$ from $\dfrac{9x+4bc}{a+x}$.

30. Subtract $m-1$ from $\dfrac{x^2+y^2}{x-y}$.

SECTION XXII.

DIVISION OF INTEGRAL AND FRACTIONAL QUANTITIES BY FRACTIONS.

Art. **69**. How many times is $\frac{3}{5}$ contained in 8? $\frac{1}{5}$ is contained in 8, 40 times, and $\frac{3}{5}$ is contained $\frac{40}{3}$ times.

How many times is $\frac{5}{9}$ contained in a? $\frac{1}{9}$ is contained in a, $9a$ times, and $\frac{5}{9}$ is contained in a, $\dfrac{9a}{5}$ times.

How many times is $\dfrac{a}{b}$ contained in c? $\dfrac{1}{b}$ is contained in c,

$b\,c$ times, and $\frac{a}{b}$ is contained in c, $\frac{b\,c}{a}$ times. This result is the same as the product of c by $\frac{b}{a}$.

How many times is $\frac{3}{8}$ contained in $\frac{4}{7}$? Reduce the fractions to a common denominator; $\frac{3}{8} = \frac{21}{56}$ and $\frac{4}{7} = \frac{32}{56}$; $\frac{21}{56}$ is contained in $\frac{32}{56}$ as many times as 21 is contained in 32, which is $\frac{32}{21}$ times.

How many times is $\frac{c}{d}$ contained in $\frac{a}{b}$? Reduce the fractions to a common denominator; $\frac{c}{d} = \frac{b\,c}{b\,d}$, and $\frac{a}{b} = \frac{a\,d}{b\,d}$; $\frac{b\,c}{b\,d}$ is contained in $\frac{a\,d}{b\,d}$, as many times as $b\,c$ is contained in $a\,d$, which is $\frac{a\,d}{b\,c}$. This result is the same as the product of $\frac{a}{b}$ by $\frac{d}{c}$.

From the preceding questions, we deduce the following

RULE FOR DIVIDING AN INTEGRAL OR FRACTIONAL QUANTITY BY A FRACTION.

Invert the divisor and then proceed as in multiplication.

Remark. When it is required to find what part one quantity is of another, make the quantity which is called the part, the dividend, and the other the divisor; also, when it is required to find the *ratio* of one quantity to another, make the quantity which is expressed first, the dividend, and the other the divisor.

Perform the following questions, taking care after inverting the divisor to omit common factors as in Art. **65**.

1. Divide m by $\frac{3}{8}$.
2. Divide $a^2\,b\,c$ by $\frac{3\,a\,b}{7\,c^2}$.
3. Divide $x^2 + y^2$ by $\frac{m^2}{4\,n}$.

4. Divide $3(a^2 - b^2)$ by $\frac{6(a-b)}{5c}$.

5. Divide $\frac{m}{n}$ by $\frac{3}{5}$.

6. Divide $\frac{a x^2}{b c}$ by $\frac{3 b c^2}{a^2 x^3}$.

7. Divide $\frac{45 x y}{7 a b}$ by $\frac{15 m x^2}{49 a^2}$.

8. Divide $\frac{11 x^2 m p}{9 a^4 b^3}$ by $\frac{99 x m^2 p^3}{4 a b^4}$.

9. Divide $\frac{x^2 + 2 x y + y^2}{4 a^2 b^2}$ by $\frac{3(x+y)}{8 a^2 b^4}$.

10. Divide $\frac{4x+2}{3}$ by $\frac{2x+1}{5x}$.

11. Divide $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$.

12. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$.

13. $\frac{a^2 b c}{4m}$ is what part of $\frac{a^3 b^2 c}{16 m^2}$?

14. $\frac{5 x^4 y^4}{17 a m^3}$ is what part of $\frac{35 x y^5}{34 a^3 m^2}$?

15. $\frac{3(x+y)}{7x^2}$ is what part of $\frac{9(x^2-y^2)}{14 m}$?

16. What is the ratio of $\frac{7bc + 21 b^2 c}{4xy}$ to $\frac{14 b^2 c^2 + 42 b c}{9 x^3 y^3 m}$?

17. What is the ratio of $\frac{155 a^2 b^2}{99 m p^3}$ to $\frac{65 a b^3}{66 m^2 p + 3 m p^2}$?

18. What is the ratio of $\frac{10(m+p)}{7 b c x^4}$ to $\frac{55(m^2-p^2)}{56 b^2 c^2 x}$?

19. What is the ratio of $\frac{4 h^3 n^3 x}{m^2-1}$ to $\frac{20 h n x^2}{m+1}$?

20. What is the ratio of $\frac{3(x^2 + 2xy + y^2)}{73 b c}$

to $\frac{12(x^3 + 3x^2 y + 3 x y^2 + y^3)}{365 b^2 c^3}$?

SECTION XXIII.

LITERAL EQUATIONS.

Art. **70**. Let the learner find the value of x in the following equations.

1. $\frac{bc-x}{a-2d}=\frac{3x+4m}{b+c}$. Multiplying by $a-2d$, we have $bc-x=\frac{3ax+4am-6dx-8dm}{b+c}$.

Multiplying by $b+c$,

$$b^2c-bx+bc^2-cx=3ax+4am-6dx-8dm.$$

Transposing all the terms containing x into the first member, and the others into the second,

$$6dx-bx-cx-3ax=4am-b^2c-bc^2-8dm.$$

Separating the first member into factors, one of which shall be x,

$$(6d-b-c-3a)x=4am-b^2c-bc^2-8dm.$$

Here x is taken $6d-b-c-3a$ times; that is, the factor $6d-b-c-3a$ is the coëfficient of x.

Dividing by this coëfficient,

$$x=\frac{4am-b^2c-bc^2-8dm}{6d-b-c-3a}; \text{ or}$$

$$x=\frac{b^2c+bc^2-4am+8dm}{b+c+3a-6d};$$ for it is evident that the signs of both numerator and denominator may be changed without affecting the value of the fraction, since $\frac{+ab}{+b}=+a$, and $\frac{-ab}{-b}=+a$. Or we might have changed the signs of all the terms in the equation, previous to separating the first member into factors.

2. $\frac{x-a}{b}=\frac{bc-cx}{a}$.

3. $\frac{a}{b}-\frac{c-x}{d}=3c-mx$.

4. $\frac{3b-4x}{b+c}=\frac{cm-dx}{b-2c}$.

5. $\frac{m^2x+bc}{2b-3x}=\frac{a+4c}{5}$.

6. $\frac{a^2x}{b-c}-dc=bx-ac$.

7. $\frac{ad^2+ax^2}{dx}=\frac{ac+ax}{d}$.

8. $\frac{ax-75}{b-x}-3a=\frac{7bc-33}{2d}$.

9. $\frac{3a-4b}{7-2x}=\frac{7b-3a}{3-x}$.

SECTION XXIV.

EQUATIONS OF THE FIRST DEGREE WITH TWO UNKNOWN QUANTITIES.

Art. 71. The problems of the first six sections involved only one unknown quantity. When a question involves several unknown quantities, there must always be as many conditions given, and, consequently, *there must result as many different equations, as there are unknown quantities.*

1. A man bought 3 barrels of cider and 2 barrels of beer for $14; and, at the same rate, 5 barrels of cider and 3 barrels of beer for $22. Required the price of each per barrel.

Let $x=$ the price of the cider per barrel,
and $y=$ the price of the beer per barrel.

Then, from the conditions of the question,

(1) $3x + 2y = 14;$
(2) $5x + 3y = 22.$ Multiply the 1st by 3 and the 2d by 2,
(3) $9x + 6y = 42;$
(4) $10x + 6y = 44.$ Subtract the 3d from the 4th,
$10x + 6y - 9x - 6y = 44 - 42;$ reduce,
$x = \$2.$ Substitute 2 for x in the 1st,
$6 + 2y = 14;$ transpose, reduce and divide,
$y = \$4.$ Ans. Cider \$2; beer \$4 per barrel.

We might have multiplied the 1st equation by 5, the 2d by 3, and then subtracted one result from the other. This would have given an equation without x, from which we could have found the value of y. Then if the value of y had been substituted in one of the preceding equations, the value of x could have been found.

2. A and B together have \$40, and if 3 times B's money be subtracted from twice A's, the remainder will be \$5. How much money has each?

Let $x =$ A's money, and $y =$ B's. Then,

(1) $x + y = 40;$
(2) $2x - 3y = 5.$ Multiply the 1st by 3,
(3) $3x + 3y = 120;$ add the 2d and 3d,
$5x = 125;$ hence, $x = \$25$, A's money.
Substituting 25 for x in the 1st,
$25 + y = 40;$ hence, $y = \$15$, B's money.

We might have multiplied the 1st equation by 2, and subtracted the 2d from the result, which would have given an equation without x.

3. A market woman sold 4 melons and 6 peaches for 60 cents, and, at the same rate, 6 melons and 15 peaches for 102 cents. Required the price of a melon and that of a peach.

Let $x =$ the price of a melon,
and $y =$ the price of a peach. Then,
(1) $4x + 6y = 60;$
(2) $6x + 15y = 102.$

The coëfficients of y in the two equations will be alike, if the 1st be multiplied by 5 and the 2d by 2; or the coëfficients of x will be alike, if the 1st be multiplied by 3 and the 2d by 2. But the best way, in this question, is to divide the 1st by 2 and the 2d by 3, which gives

$$\left.\begin{aligned}(3)\quad 2x+3y&=30;\\(4)\quad 2x+5y&=34.\end{aligned}\right\}\text{ Subtract the 3d from the 4th,}$$

$2y=4$, and $y=2$ cents, price of a peach.
Substitute 2 for y in the 3d,
$2x+6=30$, from which $x=12$ cents, price of a melon.

4. Says A to B, $\frac{1}{3}$ of my money and \$10 is equal to $\frac{1}{2}$ of yours; yes, says B, but $\frac{1}{4}$ of my money and \$10 is equal to $\frac{2}{3}$ of yours. How much money has each?

Let $x=$ A's money, and $y=$ B's. Then,

$$\left.\begin{aligned}(1)\quad \frac{x}{3}+10&=\frac{y}{2};\\(2)\quad \frac{y}{4}+10&=\frac{2x}{3}.\end{aligned}\right\}\text{ Remove the denominators,}$$

$$\left.\begin{aligned}(3)\quad 2x+\ 60&=3y;\\(4)\quad 3y+120&=8x.\end{aligned}\right\}\text{ By transposition,}$$

$$\left.\begin{aligned}(5)\quad 2x-3y&=-\ 60;\\(6)\quad 3y-8x&=-120.\end{aligned}\right\}\text{ Add the 5th and 6th,}$$

$-6x=-180$; change the signs,
$6x=180$; hence, $x=\$30$, A's money.
Substitute 30 for x in the 3d,
$60+60=3y$; from which $y=\$40$, B's money.

We see, that, in the preceding problems, the conditions, in each case, give rise to two distinct equations, which may be called the *original equations;* the others which follow, are deduced from these, or are mere modifications of them.

From the two original equations containing two unknown quantities, we obtained one with only one unknown quantity. This is called *eliminating* the unknown quantity, which this new equation does not contain. Thus, in the solution of the first

question in this article, we eliminated y, and thus obtained an equation with x only and known numbers.

We perceive, moreover, that when the quantity to be eliminated, is in the corresponding members of the two equations, that is, either in the first or second members of both, and is found in only one term of each, the following rule will enable us to effect the elimination.

FIRST METHOD OF ELIMINATION.

Art. **72**. *Multiply or divide the equations, if necessary, so as to make the coëfficients of the quantity to be eliminated the same in the two equations; then subtract one of the resulting equations from the other, if the signs of the terms containing this quantity are alike in both equations, or add them together, if the signs are different.*

In applying this rule, the equations should first be freed from fractions, if they contain any, and it is advisable to transpose all the unknown terms into the first members; moreover, if the unknown quantity to be eliminated, is found in several terms in one or both of the equations, these terms, in each, must be reduced to one.

The coëfficients of any letter in the two equations will be made alike, if, after the equations are prepared as prescribed above, each equation be multiplied by the coëfficient of that letter in the other equation; or, if each equation be multiplied by the number, by which the coëfficient of that letter in this equation must be multiplied, in order to produce the least common multiple of the two coëfficients of the letter to be eliminated.

For example, in the 3d question, the least common multiple of 4 and 6, the coëfficients of x, is 12, which may be produced by multiplying 4 by 3, or 6 by 2. If, therefore, the 1st equation be multiplied by 3 and the 2d by 2, the coëfficients of x will be alike.

1. A shoemaker sold 3 pairs of shoes and 4 pairs of boots for \$26; and, at the same rate, 5 pairs of shoes and 3 pairs of boots for \$25. What was the price of the shoes and the boots a pair?

Let $x =$ the price of a pair of shoes,
and $y =$ the price of a pair of boots. Then,

(1) $3x + 4y = 26;$
(2) $5x + 3y = 25.$ Transpose $4y$ in 1st and divide by 3,

(3) $x = \frac{26 - 4y}{3}$. Transpose $3y$ in 2d and divide by 5,

(4) $x = \frac{25 - 3y}{5}$.

Now, since the second members of equations 3d and 4th are each equal to x, they are equal to each other. Hence,

$\frac{25 - 3y}{5} = \frac{26 - 4y}{3}$. Multiply by 5 and 3, or 15,

$75 - 9y = 130 - 20y$; transpose, reduce and divide,

$y =$ \$5, price of a pair of boots.

Substitute 5 for y in the 3d,

$x = \frac{26 - 20}{3} =$ \$2, price of a pair of shoes.

2. What fraction is that to the numerator of which if 4 be added, the value of the fraction will be $\frac{1}{2}$; but if 7 be added to the denominator, the value will be $\frac{1}{5}$?

Let $x =$ the numerator, and $y =$ the denominator.

The required fraction then will be expressed by $\frac{x}{y}$. Hence,

(1) $\frac{x+4}{y} = \frac{1}{2};$
(2) $\frac{x}{y+7} = \frac{1}{5}.$ Multiply the 1st by y,

(3) $x + 4 = \frac{y}{2}$; transpose the 4,

(4) $x = \frac{y}{2} - 4$. Multiply the 2d by $y + 7$,

(5) $x = \frac{y + 7}{5}$. Put the two values of x equal,

$\frac{y}{2} - 4 = \frac{y + 7}{5}$; multiply by 10,

$5y-40=2y+14$; transpose, reduce and divide,
$y=18$, the denominator.
Substitute 18 for y in the 4th,
$x=\frac{18}{2}-4=5$, the numerator.
The fraction sought then is $\frac{5}{18}$.

In the solution of the last two questions, we found the value of x from each of the original equations, as if y were known; that is, we found from each an expression for x, consisting of y's and known numbers; then, by equalizing these two values of x, we obtained an equation without x. We might have eliminated y in a similar manner, and found an equation without that letter.

Hence, we have a

SECOND METHOD OF ELIMINATION.

Art. **73**. *Find the value of one of the unknown quantities, from each of the equations, as if the other unknown quantity were determined; then form a new equation by putting these two values equal to each other.*

Observe, however, that the unknown quantity itself must not be contained in any expression for its value.

1. Says A to B, give me one dollar of your money, and I shall have twice as much as you will have left; yes, says B, but give me one dollar of your money, and I shall have three times as much as you will have left. How much money has each?

Let $x=$ A's money, and $y=$ B's.

Then after B has given A \$1, A will have $x+1$, and B will have $y-1$ dollars. But if A gives B \$1, A will have $x-1$, and B, $y+1$.

Hence, from the conditions of the question,

$$\left.\begin{array}{ll}(1) & x+1=2y-2; \\ (2) & 3x-3=y+1.\end{array}\right\}$$ The first equation gives

(3) $x=2y-3$.

Now this value of x may be put instead of x in the 2d equation; but as x in the 2d is multiplied by 3, we must multiply

this value by 3, and the result, $6y - 9$, may be substituted for $3x$, in the 2d, which gives

$$6y - 9 - 3 = y + 1, \text{ from which we deduce}$$
$$y = \$2\tfrac{3}{5}, \text{ B's money.}$$

Substitute $2\frac{3}{5}$ for y in the 3d,

$$x = 5\tfrac{1}{5} - 3 = \$2\tfrac{1}{5}, \text{ A's money.}$$

2. The mast of a vessel consists of two parts; $\frac{1}{3}$ of the lower part, added to $\frac{1}{6}$ of the upper part, makes 28 feet; and 5 times the lower part, diminished by 6 times the upper part, is equal to 12 feet. Required the length of each part.

Let $x =$ the lower, and $y =$ the upper part. Then,

$$\left.\begin{array}{ll} (1) & \frac{x}{3} + \frac{y}{6} = 28; \\ (2) & 5x - 6y = 12. \end{array}\right\} \text{ The 1st gives}$$

$$(3) \quad y = 168 - 2x.$$

Multiply this value of y by 6, and substitute the result for $6y$ in the 2d; but, as $6y$ has the sign —, the value of $6y$ must be subtracted, that is, its signs must be changed when the substitution is made. Hence, we have

$$5x - 1008 + 12x = 12, \text{ which gives}$$
$$x = 60 \text{ feet, the lower part.}$$

Substitute 60 for x in the 3d,

$$y = 168 - 120 = 48 \text{ feet, the upper part.}$$

From the solution of the foregoing questions, we deduce a

THIRD METHOD OF ELIMINATION.

Art. **74**. *Find, from one of the equations, the value of the quantity to be eliminated, as if the other unknown quantity were determined, and substitute this value in the other equation, instead of the unknown quantity itself.*

1. There are two numbers, such that if 15 times the 2d be added to the 1st, the sum will be 53; and if 3 times the 1st be added to the 2d, the sum will be 27. What are these numbers?

2. Two men talking of their money, the first says to the second, $\frac{1}{6}$ of mine and $\frac{1}{4}$ of yours make \$6; but $\frac{1}{4}$ of mine and $\frac{1}{6}$ of yours make only \$5$\frac{2}{3}$. How much money has each?

3. A farmer sells to one man 5 cows and 7 oxen for \$370, and to another, at the same rate, 10 cows and 3 oxen for \$355. Required the price of a cow and that of an ox.

4. Says A to B, give me \$4 of your money, and I shall have as much as you; yes, says B, but give me \$4 of your money, and I shall have three times as much as you. How much money has each?

5. I can buy in the market 3 bushels of potatoes and 4 bushels of corn for \$5, and, at the same rate, 6 bushels of potatoes and 7 bushels of corn for \$9. What is the price of a bushel of each?

6. A man bought some wheat at 8s. per bushel, and some rye at 5s. per bushel, to the amount of \$20; he afterwards sold, at the same rate, $\frac{1}{2}$ of his wheat and $\frac{3}{8}$ of his rye for \$9$\frac{1}{6}$. How many bushels of each did he buy, and how many of each did he sell?

7. A laborer wrought 8 days, having his son with him 6 days, and received for both \$10; he afterwards wrought 10 days, having his son with him 9 days, and received \$13. What were the daily wages of himself and son?

8. What fraction is that, whose numerator being doubled, and the denominator increased by 8, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{1}{2}$?

9. What fraction is that, whose numerator being diminished by 3, the value becomes $\frac{3}{5}$; but the denominator being diminished by 3, the value becomes $\frac{9}{7}$?

10. A man bought coffee at 12 cents and tea at 75 cents a pound, and paid for the whole \$249; the next day he disposed of $\frac{4}{5}$ of his coffee and $\frac{2}{3}$ of his tea for \$180, which was \$10·80 more than it cost him. How many pounds of each article did he buy, and how much of each did he sell?

11. A market-woman bought eggs, some at two for a cent, and some at five for three cents, and gave for the whole \$1·60; she afterwards sold them all for \$3·10, and thereby gained $\frac{1}{2}$ a cent on each egg. How many of each kind did she buy?

12. A grocer having two casks of wine, drew out 25 gallons from the smaller and 30 from the larger, and found the number of gallons remaining in the former to the number remaining in the latter as 5 to 7; he then put 10 gallons of water into each cask, and found the number of gallons of the mixture in the smaller to the number of gallons in the larger as 3 to 4. How many gallons did each cask at first contain?

13. A man would sell 4 bushels of wheat and 9 bushels of oats for 63 shillings; or he would exchange 4 bushels of wheat for 8 bushels of oats and 12 shillings in money. At what price did he estimate the wheat and oats per bushel?

14. A grocer has two casks of wine, the larger at 12s. and the smaller at 10s. per gallon, and the whole is worth £136; from the larger he draws 60 gallons, and from the smaller 20; he then mixes the remainders together, adds 40 gallons of water to the mixture, and finds it worth 9s. per gallon. How many gallons were there at first in each cask?

15. A sportsman has a fishing rod consisting of two parts; twice the upper part exceeds the lower by 5 feet; moreover, 3 times the lower part added to 4 times the upper part, exceeds twice the whole length of the rod by 55 feet. What is the length of each part?

16. A owes \$600, and B \$800; but neither has sufficient to pay his debts. Says A to B, lend me $\frac{1}{7}$ of your money, and I shall be enabled to discharge my debts; yes, says B, but lend me $\frac{1}{5}$ of your money, and I can discharge mine. How much money has each in possession?

SECTION XXV.

EQUATIONS OF THE FIRST DEGREE WITH SEVERAL UNKNOWN QUANTITIES.

Art. **75.** 1. A drover bought at one time an ox, a cow and a calf for $65; at another, and at the same rate, two oxen, three cows and a calf for $145; and, at a third time, three oxen, two cows and a calf for $165. Required the price of an ox, a cow, and a calf.

Let $x =$ the price of an ox,
$y =$ the price of a cow,
and $z =$ the price of a calf. Then,

$$\left.\begin{array}{ll}(1) & x+y+z=65; \\ (2) & 2x+3y+z=145; \\ (3) & 3x+2y+z=165.\end{array}\right\}$$

Here we have three distinct equations, containing three different unknown quantities; and the first step in the solution, is, to deduce from them two equations containing only two different unknown quantities. Let us eliminate z, that is, obtain two equations without z.

First method. The coëfficients of z being alike in the three equations, by subtracting the 1st from the 2d, also the 2d from the 3d, we have

$$\left.\begin{array}{ll}(4) & x+2y=80; \\ (5) & x-y=20.\end{array}\right\}$$

Since equations 4th and 5th do not contain z, we may obtain from them the values of x and y. Subtract the 5th from the 4th,

$3y=60$; hence, $y=\$20$, price of a cow.
Substitute 20 for y in the 5th,
$x-20=20$; hence, $x=\$40$, price of an ox.
Substitute the values of x and y in the 1st,
$40+20+z=65$; hence, $z=\$5$, price of a calf.

Second method. Resume the original equations,

$$\left.\begin{array}{ll}(1) & x+\ \ y+z=\ 65;\\(2) & 2x+3y+z=145;\\(3) & 3x+2y+z=165.\end{array}\right\}$$

Deduce the value of z from each equation, as if x and y were known,

$$\left.\begin{array}{ll}(4) & z=\ 65-x-y;\\(5) & z=145-2x-3y;\\(6) & z=165-3x-2y.\end{array}\right\}$$

Put equal to each other the values of z in the 4th and 5th; also, the values of z in the 5th and 6th,

$$\begin{array}{ll}(7) & 65-x-y=145-2x-3y;\\(8) & 145-2x-3y=165-3x-2y.\end{array}$$

Transpose and reduce in the 7th and 8th,

$$\left.\begin{array}{ll}(9) & x+2y=80;\\(10) & x-y=20.\end{array}\right\}$$

Find the value of x in the 9th, also in the 10th,

$$\left.\begin{array}{ll}(11) & x=80-2y;\\(12) & x=20+\ y.\end{array}\right\}$$

Put the values of x in the 11th and 12th equal,

$20+y=80-2y$; hence, $y=$ \$20, price of a cow.

Substitute 20 for y in the 12th,

$x=20+20=$ \$40, price of an ox.

Substitute 40 for x and 20 for y in the 4th,

$z=65-40-20=$ \$5, price of a calf.

Third method. Take the original equations,

$$\left.\begin{array}{ll}(1) & x+\ \ y+z=\ 65;\\(2) & 2x+3y+z=145;\\(3) & 3x+2y+z=165.\end{array}\right\}$$

Deduce the value of z from the 1st, as if x and y were known,

$$(4)\quad z=65-x-y.$$

Substitute this value of z in the 2d and 3d,

$$\begin{array}{ll}(5) & 2x+3y+65-x-y=145;\\(6) & 3x+2y+65-x-y=165.\end{array}$$

Transpose and reduce in the 5th and 6th,

$$\left.\begin{array}{ll}(7) & x+2y=\ 80; \\ (8) & 2x+y=100.\end{array}\right\}$$

Deduce the value of x from the 7th,

$$(9) \quad x=80-2y.$$

Double this value of x, and substitute the result in the 8th,

$160-4y+y=100$; which gives $y=\$20$, price of a cow.

Substitute 20 for y in the 9th, and we have

$x=\$40$, price of an ox.

Substitute 40 for x and 20 for y in the 4th, and we have

$z=\$5$, price of a calf.

2. There are three men, A, B and C, whose ages are such, that if $\frac{2}{3}$ of A's, $\frac{7}{8}$ of B's and $\frac{3}{4}$ of C's be added, the sum will be 70 years; if twice A's be added to B's, the sum will be 5 times C's; and if $\frac{1}{3}$ of A's be subtracted from $\frac{1}{2}$ of B's, the remainder will be $\frac{1}{2}$ of C's. Required their ages.

Let x, y and z represent the respective ages of A, B and C. Then,

$$\left.\begin{array}{ll}(1) & \frac{2}{3}x+\frac{7}{8}y+\frac{3}{4}z=70; \\ (2) & 2x+\ \ y=5z; \\ (3) & \frac{1}{2}y-\frac{1}{3}x=\frac{1}{2}z.\end{array}\right\}$$

Remove the denominators in the 1st and 3d, and bring down the 2d,

$$\left.\begin{array}{ll}(4) & 16x+21y+18z=1680; \\ (2) & 2x+y=5z; \\ (5) & -2x+3y=3z.\end{array}\right\}$$

First method. Transpose all the unknown quantities into the first members, and bring down the 4th,

$$\begin{array}{ll}(4) & 16x+21y+18z=1680; \\ (6) & 2x+y-5z=0; \\ (7) & -2x+3y-3z=0.\end{array}$$

To eliminate x, add the 6th and 7th; also, add the 4th to 8 times the 7th,

$$\left.\begin{array}{ll}(8) & 4y-8z=0; \\ (9) & 45y-6z=1680.\end{array}\right\}$$

To eliminate z from these two last equations, divide the 8th by 4 and the 9th by 3,

$$\left.\begin{array}{ll}(10) & y-2z=0; \\ (11) & 15y-2z=560.\end{array}\right\}$$

Subtract the 10th from the 11th,

$14y=560$; hence, $y=40$ years, B's age.

Substitute 40 for y in the 10th,

$40-2z=0$; hence, $z=20$ years, C's age.

Substitute 40 for y and 20 for z in the 2d,

$2x+40=100$; from which, $x=30$ years, A's age.

Second method. Take the original equations cleared of fractions, that is, the 4th, 2d and 5th, inverting the order of the members in the 2d and 5th,

$$\left.\begin{array}{ll}(4) & 16x+21y+18z=1680; \\ (2) & 5z=2x+y; \\ (5) & 3z=-2x+3y.\end{array}\right\}$$

Deduce the value of z from each of these,

$$(6)\quad z=\frac{1680-16x-21y}{18};$$

$$(7)\quad z=\frac{2x+y}{5};$$

$$(8)\quad z=\frac{-2x+3y}{3}.$$

Put equal to each other the values of z in the 6th and 8th; also, the values of z in the 7th and 8th,

$$\left.\begin{array}{ll}(9) & \dfrac{-2x+3y}{3}=\dfrac{1680-16x-21y}{18}; \\ (10) & \dfrac{2x+y}{5}=\dfrac{-2x+3y}{3}.\end{array}\right\}$$

Clear the 9th and 10th of fractions, transpose, reduce, and find the value of x from each,

$$(11)\quad x=\frac{1680-39y}{4};$$

$$(12)\quad x=\frac{3y}{4}$$

Put the values of x in the 11th and 12th equal,

$$\frac{3y}{4} = \frac{1680 - 39y}{4}; \text{ hence, } y = 40 \text{ years, B's age.}$$

Substitute 40 for y in the 12th,

$x = 30$ years, A's age.

Substitute 30 for x and 40 for y in the 7th,

$$z = \frac{60 + 40}{5} = \frac{100}{5} = 20 \text{ years, C's age.}$$

Third method. Resume the 4th, 2d and 5th equations,

$$\left.\begin{array}{ll}(4) & 16x + 21y + 18z = 1680; \\ (2) & 5z = 2x + y; \\ (5) & 3z = -2x + 3y.\end{array}\right\}$$

Deduce the value of z from the 5th,

$$(6) \quad z = \frac{-2x + 3y}{3}.$$

Substitute this value in the 4th and 2d,

$$\left.\begin{array}{ll}(7) & 16x + 21y - 12x + 18y = 1680; \\ (8) & \dfrac{-10x + 15y}{3} = 2x + y.\end{array}\right\}$$

Clear the 8th of fractions, and reduce the 7th and 8th,

$$\left.\begin{array}{ll}(9) & 4x + 39y = 1680; \\ (10) & 4x = 3y.\end{array}\right\}$$

Deduce the value of x from the 10th,

$$(11) \quad x = \frac{3y}{4}.$$

Substitute this value in the 9th,

$3y + 39y = 1680$; hence, $y = 40$ years, B's age.

Substitute 40 for y in the 11th,

$x = \frac{3}{4}$ of $40 = 30$ years, A's age.

Substitute 30 for x and 40 for y in the 6th,

$$z = \frac{-60 + 120}{3} = \frac{60}{3} = 20 \text{ years, C's age.}$$

3. A boy bought of one man 3 apples, 2 peaches, 4 pears and 2 oranges for 22 cents; of a second, at the same rate, 2 apples, 3 peaches, 2 pears and 4 oranges for 24 cents; of a third, 5 apples, 1 peach, 6 pears and 10 oranges for 36 cents; and of a fourth, 4 apples, 3 peaches, 2 pears and 8 oranges for 32 cents. What did he give apiece for each?

Let x represent the price of an apple, y that of a peach, z that of a pear, and u that of an orange. Then,

$$\left.\begin{array}{ll}(1) & 3x+2y+4z+2u=22; \\ (2) & 2x+3y+2z+4u=24; \\ (3) & 5x+y+6z+10u=36; \\ (4) & 4x+3y+2z+8u=32.\end{array}\right\}$$

Let us eliminate y, that is, obtain three equations without y. The three equations below may be found as follows. To obtain the 5th, multiply the 3d by 2, and subtract the 1st from the result. To obtain the 6th, subtract the 2d from the 4th. To obtain the 7th, multiply the 3d by 3, and subtract the 4th from the result.

$$\left.\begin{array}{ll}(5) & 7x+8z+18u=50; \\ (6) & 2x+4u=8; \\ (7) & 11x+16z+22u=76.\end{array}\right\}$$

Now let us eliminate z from the last three equations. But since the 6th does not contain z, we may divide it by 2 and place the result below. To obtain the 9th, multiply the 5th by 2 and subtract the 7th from the result.

$$\left.\begin{array}{ll}(8) & x+2u=4; \\ (9) & 3x+14u=24.\end{array}\right\}$$

To eliminate x from the last two equations, multiply the 8th by 3 and subtract the result from the 9th.

$8u=12$; hence, $u=1\frac{1}{2}$ cent, price of an orange.

Substitute the known value of u in the 8th, and we have

$x=1$ cent, price of an apple.

Substitute the values of x and u in the 5th, and we have

$z=2$ cents, price of a pear.

Substitute the values of x, z and u, in the 3d, and we have

$y = 4$ cents, price of a peach.

Let the learner solve this question by the second and third methods of elimination.

From the solution of the foregoing questions, it is easy to see, that the three modes of elimination given in the last section, may be extended to any number of equations.

To apply the *first method* for the purpose of eliminating a particular quantity from *several* equations, it is only necessary to operate upon these equations taken two and two.

In applying the *second method* to several equations, we must find, from each equation that contains it, the value of the unknown quantity to be eliminated, and then put any two of these expressions for its value equal to each other.

To extend the *third method*, we must, after having found from one of the equations the value of the unknown quantity to be eliminated, substitute this value in every other equation that contains this unknown quantity.

If a question involves five unknown quantities, and gives rise to five different equations, we should first deduce from them four equations with only four different unknown quantities; secondly, from these we should deduce three equations with only three unknown quantities; thirdly, from these three, two with only two unknown quantities; and, finally, from these two, one equation with only one unknown quantity, from which the value of this unknown quantity might be determined.

Or if six equations containing six unknown quantities, were given, we should first obtain from them five containing only five unknown quantities, and then proceed as before; and so on, if a still greater number of equations were given.

If either of the equations does not contain the unknown quantity to be eliminated, that equation may be put aside to be placed with the next set of equations, viz: those which contain one less unknown quantity.

Either of the methods of elimination may be used, but the first will generally be found most convenient, because it does not give rise to fractions. It will, however, be useful for the learner to perform every example by each method, in order to familiarize him with the process, and enable him to judge which will be best in any particular case. It is not necessary that the same mode of elimination should be pursued throughout the solution of a question, but either of them may be resorted to whenever it shall seem the most convenient.

4. A merchant bought at one time 4 barrels of flour, 3 barrels of rice, and 2 boxes of sugar for \$72; at another, 2 barrels of flour, 5 barrels of rice, and 3 boxes of sugar for \$84; and at a third time, 5 barrels of flour, 9 barrels of rice, and 8 boxes of sugar for \$187. What were the flour and rice per barrel, and what was the sugar per box?

5. There are three numbers, such that if 3 times the 2d be subtracted from 4 times the 1st, and twice the 3d be added to the remainder, the result will be 9; if twice the 1st and 5 times the 2d be added, and from the sum 3 times the 3d be subtracted, the remainder will be 4; and if 5 times the 1st and 6 times the 2d be added, and from the sum twice the 3d be subtracted, the remainder will be 18. What are these numbers?

6. Three boys, A, B and C, counting their money, it was found that twice A's added to B's and C's, would make \$5·25; that if A's and twice B's were added, and from the sum C's were subtracted, the result would be \$3·00; and the three together had \$3·25. How much money had each?

7. Three men owed together a debt of \$1000, but neither of them had sufficient money to pay the whole alone. The first could pay the whole, if the second and third would give him $\frac{5}{14}$ of what they had; the second could pay it, if the first and third would give him $\frac{4}{13}$ of what they had; and the third could pay it, if the first and second would give him $\frac{2}{11}$ of what they had. How much money had each?

8. The ages of three men, A, B and C, are such, that $\frac{1}{2}$ of A's, $\frac{1}{4}$ of B's and $\frac{1}{3}$ of C's make 80 years; $\frac{1}{3}$ of A's, $\frac{1}{2}$ of B's and $\frac{1}{5}$ of C's make 78 years; and $\frac{1}{6}$ of A's, $\frac{1}{8}$ of B's and $\frac{1}{6}$ of C's make 35 years. Required the age of each.

9. Four men could earn together in one day 26 shillings. If the 1st wrought 6 days, the 2d 8, the 3d 9, and the 4th 12, they would all earn 237s.; if the 1st wrought 2 days, the 2d 5, the 3d 7, and the 4th 9, they would earn 161s.; and if the 1st wrought 4 days, the 2d 3, the 3d 2, and the 4th 1, they would earn 60s. What were the daily wages of each?

10. A merchant had four kinds of tea, marked A, B, C and D. If he mixed 7 pounds of A, 10 of B, 12 of C and 18 of D, the whole mixture would be worth \$22·90; if he mixed 4 pounds of A, 5 of B, 8 of C and 11 of D, the whole would be worth \$13·80; if he mixed 4 pounds of A and 9 of C, the mixture would be worth \$5·70; and if he mixed 18 pounds of A, 12 of B and 36 of D, the mixture would be worth \$31·80. What was each kind worth a pound?

11. I find that I can buy in the market 1 bushel of wheat, 2 bushels of rye, 3 of barley, 4 of oats and 6 of potatoes for \$12; 3 bushels of wheat, 4 of rye, 8 of barley, 3 of oats, and 4 of potatoes for \$24$\frac{1}{2}$; 5 bushels of wheat, 2 of rye, 10 of barley, 6 of oats and 8 of potatoes for \$32; and 8 bushels of wheat, 7 of rye, 6 of barley, 5 of oats and 4 of potatoes for \$35$\frac{1}{2}$; moreover, that a bushel of wheat and a bushel of oats cost as much as a bushel of rye and a bushel of barley. Required the price of a bushel of each.

SECTION XXVI.

NUMERICAL SUBSTITUTION OF ALGEBRAIC QUANTITIES

Art. **76.** Find the numerical value of the following quantities, when $a=5$, $b=6$, $c=7$, and $d=10$.

1. ab. Ans. $6 . 5=30$.
2. a^2bc. Ans. $5^2 . 6 . 7=1050$.
3. $abcd$.
4. a^2bcd.
5. ab^2cd.
6. a^3b^2.
7. acd^3.
8. $\frac{ab}{cd}$.
9. $\frac{abc}{d}$.
10. $\frac{cd^2}{a}$.
11. $\frac{a^2b^2}{c^2d}$.
12. $\frac{a}{bd}$.
13. $a+b+c$.
14. $ab+c$.
15. $a+bc$.
16. $ab+cd$.
17. $abc+d$.
18. $a+bcd$.
19. $2ab+3d$.
20. $a^2+b+c+d$.
21. ab^2+cd^2.
22. $a+b+d^3$.
23. $a+b-c$.
24. $a+b-c-d$.
25. $a^2-b-c+d^2$.
26. $(a+b+c)d$.
27. $(b+c+d)a$.
28. $ab(c+d)$.
29. $(a+b)(c+d)$.
30. $(a+b)(d-c)$.
31. $(a+b)(c-d)$.
32. $(a-b)(c-d)$.
33. $(b^2-a^2)(c+d)$.
34. $(a+b)(c^2+d^2)$.
35. $(b^2-a^2)(d^2-c^2)$.
36. $(a^2-b^2)(c^2-d^2)$.
37. $(a+b)^2$.
38. $(a+b)^2cd$.

Find the value of the following expressions, when $a=0$, $b=2$, $n=4$, and $m=6$

39. $a+b-n+m$
40. $3a+2b+n-m$.
41. $ab+mn$.
42. $6a-5abn$.
43. $3m-7abn$.

44. $\frac{a+b}{m-n}$.

45. $\frac{-b+a}{n-m}$.

46. $\frac{b\,m}{6+m\,a}$.

47. $\frac{a^3 b-3\,m\,n}{a-4\,n}$.

48. $\frac{m^3-4\,a\,b\,n}{a-27\,b^3}$.

49. $\frac{(a+b)(m-n)}{(m-a)^2}$.

Substitute numbers in the following equivalent expressions, showing their identity whatever numbers are put instead of the letters, observing however to give the same value to the same letter in the two members of an equation.

50. $(a+b)\,c=a\,c+b\,c$.

Suppose $a=2$, $b=3$, and $c=6$.

Then, $(a+b)\,c=(2+3)\,6=5\,.\,6=30$.

Also, $a\,c+b\,c=2\,.\,6+3\,.\,6=12+18=30$.

51. $(a+b)(a-b)=a^2-b^2$.

52. $(a+b)^2=a^2+2\,a\,b+b^2$.

53. $(a-m)^2=a^2-2\,a\,m+m^2$.

54. $a+\frac{b}{c}=\frac{a\,c+b}{c}$.

55. $\frac{m^3-1}{m-1}=m^2+m+1$.

56. $\frac{m^4-1}{m+1}=m^3-m^2+m-1$.

57. $\frac{a^3+b^3}{a+b}=a^2-a\,b+b^2$.

58. $\frac{a^3+8}{a+2}=a^2-2\,a+4$.

59. $(a+b)(a+c)=a^2+a\,(b+c)+b\,c$.

60. $(a+b)(a+c)(a+d)=a^3+a^2\,(b+c+d)+a\,(b\,c+b\,d+c\,d)+b\,c\,d$.

61. $\frac{b}{a+b}+\frac{a-b}{b}=\frac{a^2}{a\,b+b^2}$.

62. $\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{2\,(a^2+b^2)}{a^2-b^2}$.

63. $\dfrac{a^2-2a+1}{a^2-1}=\dfrac{a-1}{a+1}$.

64. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{4ab}{a^2-b^2}=0$.

SECTION XXVII.

GENERALIZATION.

Art. 77. In the problems of the first six, as also in those of the 24th and 25th sections, letters have been used to represent unknown quantities only, and the results, expressed in definite numbers, correspond to the particular questions only, from which they were derived.

But in *pure* algebra, letters may also represent known quantities, or they may be used indefinitely, and afterwards any numbers may be substituted in their place. Also the results of pure algebra, which are called *formulæ*, show by what operations they were obtained, and furnish rules for the solution of all questions of the same kind.

1. Two men, A and B, are to share \$420, of which B is to have 3 times as much as A. Required the share of each.

Here the object is to separate \$420 into two parts, such that one shall be 3 times as great as the other.

$$
\begin{aligned}
\text{Let}\quad x &= \text{A's share};\\
\text{then,}\quad 3x &= \text{B's share.}\\
\text{Hence,}\quad x+3x &= 420;\\
x &= \$105,\ \text{A's share};\\
3x &= \$315,\ \text{B's share.}
\end{aligned}
$$

Instead of 420, in this question, put the letter a; then the problem will be, to separate the number a into two parts, one of which shall be 3 times the other.

Representing the shares as before, we have

$x+3x=a$. Hence,

$$\left.\begin{array}{l} x=\frac{a}{4}, \text{ A's share,} \\ 3x=\frac{3a}{4}, \text{ B's share.} \end{array}\right\} \text{General formulæ.}$$

If we now put \$420 instead of a in these formulæ, we have

$$\left.\begin{array}{l} x=\frac{420}{4}=\$105, \text{ A's share,} \\ 3x=\frac{3\,.\,420}{4}=\$315, \text{ B's share.} \end{array}\right\} \text{Particular answers.}$$

We perceive, from the general formulæ, that one part is a fourth, and the other three-fourths, of the number to be divided, without regard to the particular value of that number.

Let the learner put other numbers instead of a in the formulæ, and find the two parts. Any number divisible by 4 will give whole numbers for these parts.

2. A father left by his will \$4500 to be divided between his son and daughter, with the condition that the son was to receive \$500 more than the daughter. What was the share of each?

In this problem it is required to separate \$4500 into two parts, such that one shall exceed the other by \$500. Instead of \$4500, let us suppose that the number to be separated into parts is indefinite, and that it is represented by a; also, that b represents the excess of the greater part above the less. Then the problem is, to separate the number a into two parts, such that the greater shall exceed the less by b.

Let $x=$ the less part; then,

$x+b=$ the greater part. Hence,

$x+x+b=a$. Reduce and transpose b,

$2x=a-b$; divide by 2,

$x=\frac{a}{2}-\frac{b}{2}=\frac{a-b}{2}$, *the less part.*

To obtain the greater we add b to the less, and we have

$$x+b=\frac{a}{2}-\frac{b}{2}+b.$$ Change b to halves,

$$x+b=\frac{a}{2}-\frac{b}{2}+\frac{2b}{2};$$ reduce,

$$x+b=\frac{a}{2}+\frac{b}{2}=\frac{a+b}{2},$$ *the greater part.*

If we examine the formulæ for the two parts, and recollect that a and b may stand for any numbers, provided that b is less than a, we see that they furnish the following rule for separating a quantity into two parts.

The less part is found by subtracting half the excess of the greater above the less from half the number to be separated; or, by subtracting the excess of the greater above the less from the number to be separated, and dividing the remainder by 2.

The greater part is found by adding half the excess of the greater above the less to half the number to be separated; or, by adding the excess of the greater above the less to the number to be separated, and dividing the sum by 2.

Let the learner separate each of the following numbers into two parts by means of the formulæ, or by following the rule.

	Numbers to be separated.	*Excess of one part over the other.*
3.	150	30.
4.	230	50.
5.	1200	120.
6.	27	5.
7.	35	3.
8.	70	3.
9.	$47\frac{1}{2}$	13.
10.	99	$33\frac{1}{3}$.

11. Separate a number a into three parts, such that the *mean* shall exceed the least by b, and the *greatest* shall exceed the *mean* by c.

Let $x=$ the least part;
then $x+b=$ the mean part;
and $x+b+c=$ the greatest part.

Hence, $x+x+b+x+b+c=a$. Reducing,

$3x+2b+c=a$; transposing,

$3x=a-2b-c$; dividing,

$$x=\frac{a}{3}-\frac{2b}{3}-\frac{c}{3}=\frac{a-2b-c}{3}, \text{ the least part.}$$

Adding b to the least, we have

$$x+b=\frac{a}{3}-\frac{2b}{3}-\frac{c}{3}+b=\frac{a}{3}-\frac{2b}{3}-\frac{c}{3}+\frac{3b}{3}$$

$$=\frac{a}{3}+\frac{b}{3}-\frac{c}{3}=\frac{a+b-c}{3}, \text{ the mean part.}$$

Adding c to the mean, we have

$$x+b+c=\frac{a}{3}+\frac{b}{3}-\frac{c}{3}+c=\frac{a}{3}+\frac{b}{3}-\frac{c}{3}+\frac{3c}{3}$$

$$=\frac{a}{3}+\frac{b}{3}+\frac{2c}{3}=\frac{a+b+2c}{3}, \text{ the greatest part.}$$

Let the learner translate these formulæ into rules, recollecting that a represents the number to be separated, b the excess of the mean above the least, and c the excess of the greatest above the mean.

12. A man bought sugar at a cents, flour at b cents, and coffee at c cents per pound, and the whole amounted to d cents. How many pounds of each did he buy, if he bought the same quantity of each?

Let $x=$ the number of pounds each.
Then, $ax+bx+cx=d$.

Separating the 1st member into factors, one of which is x,

$(a+b+c)x=d$. Dividing by the coëfficient of x,

$$x=\frac{d}{a+b+c}, \text{ the number of pounds of each.}$$

This formula may be translated into the following rule, viz:

Divide the price of the whole by the sum of the prices of a pound of each sort; the quotient will be the number of pounds of each.

If in the formula we substitute the numbers 7, 6, 10 and 92, for a, b, c, and d respectively, we have

$$x=\frac{92}{7+6+10}=\frac{92}{23}=4 \text{ lbs., particular answer.}$$

13. A farmer found he had a times as many cows as oxen, and b times as many sheep as cows, and that his whole stock amounted to the number c. Required the number of each.

Let $x=$ the number of oxen;
then $ax=$ the number of cows;
and $abx=$ the number of sheep.
Hence, $x+ax+abx=c$.

Here x is taken $1+a+ab$ times; therefore,

$(1+a+ab)x=c$. Dividing by the coëfficient of x,

$$x=\frac{c}{1+a+ab}, \text{ number of oxen.}$$

$$ax=\frac{c}{1+a+ab}\times a, \text{ number of cows.}$$

$$abx=\frac{c}{1+a+ab}\times ab, \text{ number of sheep.}$$

If 3 be put for a, 4 for b, and 128 for c, in these formulæ, we have

the number of oxen $=\dfrac{128}{1+3+12}=\frac{128}{16}=8$;
the number of cows $=3.8=24$;
the number of sheep $=4.3.8=96$.

14. What will be the particular answers in the preceding question, if $a=5$, $b=7$, and $c=369$?

15. Two men had engaged to perform a certain piece of work; the first could do it alone in a days, and the second in b days. How long would it take both working together to do it?

Let $x=$ the number of days in which both would do it.

Then, as the first could do the whole in a days, in 1 day he would do $\frac{1}{a}$ of it; and, as the second could do the whole in b

days, in 1 day he would do $\frac{1}{b}$ of it; so that both would perform $\frac{1}{a}+\frac{1}{b}$ of it in 1 day, and in x days, $\frac{x}{a}+\frac{x}{b}$. But in x days, we have supposed that they would perform the whole. Hence,

$\frac{x}{a}+\frac{x}{b}=1$, piece of work.

Multiplying by a and b,

$bx+ax=ab$; or,

$(b+a)x=ab$; dividing by $b+a$,

$x=\frac{ab}{b+a}$. Answer.

From this formula we derive the following rule for any similar case, in which two workmen are employed.

Divide the product of the numbers expressing the times in which each would perform it, by the sum of those numbers.

Let the learner find the answers to the following questions, by means of the preceding formula.

16. If A could perform a piece of work in 6 days, and B could perform the same in 5 days, how long would it take both together to perform it?

17. By a pipe A, a certain cistern will be filled with water in $5\frac{1}{2}$ hours, and by another pipe B, it will be filled in $8\frac{1}{4}$ hours; in what time will it be filled, if water flow through both pipes at the same time?

18. Let it be proposed to find a rule for dividing the gain or loss in partnership, or, as it is commonly called, the *rule of fellowship*. First, take a particular case.

Three men traded in company and put in stock in the following proportions, viz: A put in \$5 as often as B put in \$3 and as often as C put in \$2. The company gained \$650. Required each man's share of the gain.

Let $x =$ A's share. Then, since B furnished $\frac{3}{5}$ as much stock as A, he must have $\frac{3}{5}$ as much gain; therefore,

$\frac{3x}{5} =$ B's share. In like manner,

$\frac{2x}{5} =$ C's share. Hence,

$x + \frac{3x}{5} + \frac{2x}{5} = 650.$ Multiplying by 5,

$$5x + 3x + 2x = 3250, \text{ or}$$
$$10x = 3250;$$
$$x = \$325, \text{ A's share.}$$
$$\frac{3x}{5} = \$195, \text{ B's share.}$$
$$\frac{2x}{5} = \$130, \text{ C's share.}$$

To generalize this question, suppose that A put in m as often as B put in n and as often as C put in p dollars, and that they gained a dollars. Then B puts in $\frac{n}{m}$, and C $\frac{p}{m}$ as much as A.

Let $x =$ A's gain; then,

$\frac{nx}{m} =$ B's gain, and

$\frac{px}{m} =$ C's gain. Hence,

$x + \frac{nx}{m} + \frac{px}{m} = a.$ Multiplying by m,

$mx + nx + px = ma$; or,

$(m + n + p)x = ma$; dividing by the coëfficient of x,

$x = \frac{ma}{m+n+p}$, or $m \times \frac{a}{m+n+p}$, A's share.

B's share is $\frac{n}{m}$ of A's; $\frac{1}{m}$ of $m \times \frac{a}{m+n+p}$ is $\frac{a}{m+n+p}$, and $\frac{n}{m}$ of it is n times as much; therefore,

$\frac{nx}{m} = n \times \frac{a}{m+n+p}$, B's share.

In like manner, C's share is $\frac{p}{m}$ of A's; or

$$\frac{px}{m} = p \times \frac{a}{m+n+p}, \text{ C's share.}$$

By examining these formulæ, we perceive that the whole gain a is divided by $m+n+p$, the sum of the proportions of the stock furnished by all the partners, and that this quotient is multiplied by m, A's proportion, by n, B's proportion, and by p, C's proportion of the stock, to obtain their respective shares of the gain. Hence, observing that a may represent the loss as well as the gain, to find each partner's share of gain or loss, we have the following rule.

Divide the whole gain or loss by the sum of the proportions of the stock, and multiply the quotient by each partner's proportion.

This rule is applicable, whatever be the number of partners.

19. Suppose A put in **$400, B $300,** and C **$200,** and that **they gained $450.** By the preceding formulæ, or by the rule, what was each partner's share of the gain?

Remark. When the sums actually put in are given, the simplest proportions of the stocks will be found by dividing these sums by the greatest number that will divide them all without any remainder. Thus, 400, 300, and 200 are all divisible by 100; and the quotients, 4, 3, and 2, express the proportions of the stock.

20. What would be each man's loss, if A furnished $300, B $150, and C $100, the entire loss being $99?

21. What would be each man's share of $500 gained, if four partners furnished respectively $800, $600, $400, and $200?

22. A put in $200 for 6 months, B $150 for 5 months, and C $300 for 2 months. They gained $272; what was each man's share of this gain?

Remark. It is evident, that, if the stocks are employed unequal times, each partner's stock, or his proportion of the stock, must be multiplied by the number expressing the time during which it is in trade, and that then the proportions of these products must be used.

In general, known quantities are represented by the first, and unknown quantities by the last letters of the alphabet. But, in some cases, it is more convenient to use the initial letters of the names of quantities, whether known or unknown.

In the following questions relating to simple interest, let p represent the principal, r the rate, t the time, i the interest, a the amount, and d the discount. In these questions, r is supposed to be a fraction, as ·06, ·05, &c., according as the rate is 6 per cent., 5 per cent., &c., and the time is supposed to be expressed in years and fractions of a year.

23. What is the simple interest of p dollars, for t years, at r per cent.?

The principal multiplied by the rate gives the interest for one year; hence,

$rp =$ the interest for 1 year; and

$trp =$ the interest for t years. Therefore,

$$i = trp.$$

This formula gives the following rule.

To find the interest when the principal, rate, and time are known, multiply together the principal, time, and rate.

24. The principal being \$256·25, the time $4\frac{1}{2}$ years, and the rate 6 per cent., what will be the interest?

In the equation $trp = i$, provided any three of the quantities are given, the other may be found. Let the learner find the formulæ and make rules for the following general questions, and solve by the rules the particular examples subjoined.

25. The interest, time, and rate being given, to find the principal.

26. If the interest, for 7 years at 5 per cent., is \$26·25, what is the principal?

27. The interest, time, and principal given, to find the rate.

28. The interest being \$74·4711, time 6 years, and the principal \$225·67, what is the rate?

29. The interest, rate, and principal given, to find the time.

30. If the interest is \$102, rate $4\frac{1}{2}$ per cent., and the principal \$320, what is the time?

31. What is the amount of p dollars, for t years, at the rate r, simple interest?

The amount being the sum of the principal and interest, we have

$$a = p + t\,r\,p;\ \text{or},\ a = p\,(1 + t\,r).$$

This formula gives the following rule.

To find the amount, when the principal, time, and rate are known, multiply the time and rate together, add 1 *to the product, and multiply this sum by the principal.*

32. The principal being $650, rate 4½ per cent., and the time 7 years and 3 months, what is the amount by the preceding rule?

The equation, $p + t\,r\,p = a$, contains four different quantities, any three of which being known, the other may be determined.

Let the learner find formulæ and rules for the following general questions, and solve the particular examples by the rules.

33. The amount, time, and rate being given, to find the principal; that is, to find what sum of money put at interest, at a given rate, and in a specified time, will amount to a given sum.

N. B. The principal, in this case, is sometimes called the *present worth* of the amount.

34. What is the present worth of $300, due in 3 years and 4 months, the rate being 6 per cent.?

35. The amount, principal, and time given, to find the rate.

36. The amount being $405·09, principal $321·50, the time 4 years, what is the rate?

37. The amount, principal, and rate given, to find the time.

38. Amount $352, principal $275, and rate 8 per cent., required the time.

39. The amount, time, and rate given, to find the discount.

Remark. The formula for the discount may be found by subtracting the formula for the present worth from the amount a, and simplifying the result

40. Required the discount on £100, due in 3 months, the rate being 5 per cent.

41. At a given rate, in what time will a sum be doubled? In what time will it we tripled?

Remark. Take the formula for the amount, put $2p$ and $3p$ successively instead of a, and then find the value of t.

42. In what time will a sum be doubled at 6 per cent.? In what time will it be tripled?

43. In what time will a sum be doubled at 5 per cent.? In what time will it be tripled?

44. Separate the number a into two parts, one of which shall be n times the other.

45. Separate a into two parts, so that the second may be the $\frac{m}{n}$ part of the first.

46. Separate a into three parts, such that the second shall be m times, and the third n times the first.

47. Separate a into two parts, so that if one of them be divided by b, and the other by c, the sum of the quotients shall be d.

48. Separate a into two parts, so that the mth part of one shall exceed the nth part of the other by b.

49. What number is that whose mth part exceeds its nth part by p?

50. After paying away $\frac{1}{m}$ and $\frac{1}{n}$ of my money, I had a guineas left. How many guineas had I at first?

51. After paying away the $\frac{m}{n}$ and the $\frac{p}{q}$ parts of my money, I had a dollars left. How much money had I at first?

52. A and B together could do a piece of work in a days; B could do it alone in b days; in how many days could A do it alone?

53. A company at a tavern paid a shillings each; but if there had been b persons less, each would have had to pay c shillings. How many persons in the company?

54. A gentleman has 6 sons, each of whom is a years older than his next younger brother, and the eldest is b times as old as the youngest. Required their ages.

55. A person borrowed as much money as he had in his purse, and then spent a shillings; again he borrowed as much as he had left in his purse, after which he spent a shillings; he borrowed and spent, in the same manner, a third and fourth time; after the fourth expenditure he had nothing left. How much money had he at first?

56. A man agreed to work n days, with this condition, that he should receive a shillings for every day he worked, but should forfeit b shillings for every day he was idle. At the end of the time agreed on, he received a balance of c shillings. How many days did he work, and how many was he idle?

57. A gentleman gave some beggars a cents apiece and had b cents left; but if he had given them c cents apiece, he would have been obliged to borrow d cents for that purpose. How many beggars were there?

The following questions may be solved by means of two unknown quantities.

58. Said A to B, the sum of our ages is a years, and their difference is b years. Required their ages, A being the elder.

59. One pair of boots and a pairs of shoes cost b dollars; and c pairs of boots and one pair of shoes cost d dollars. Required the price of the boots and shoes a pair.

60. There are two numbers, such that if $\frac{1}{b}$ part of the second be added to the first, the sum will be a; and if $\frac{1}{c}$ part of the first be added to the second, the sum will also be a. Required the two numbers.

61. What fraction is that, to the numerator of which if a be added, the value of the fraction will become $\frac{m}{n}$; but if a be added to the denominator, the value of the fraction will be $\frac{p}{q}$?

62. What fraction is that, from the numerator of which if a be subtracted, the value of the fraction will become $\frac{m}{n}$; but if a be subtracted from the denominator, the value of the fraction will become $\frac{p}{q}$?

63. What will be the particular answer to the 61st, if $a=2$, $\frac{m}{n}=\frac{2}{3}$, and $\frac{p}{q}=\frac{4}{11}$; and what will be the particular answer to the 62d, if $a=3$, $\frac{m}{n}=\frac{8}{15}$, and $\frac{p}{q}=\frac{11}{12}$?

SECTION XXVIII.

NEGATIVE QUANTITIES AND THE INTERPRETATION OF NEGATIVE RESULTS.

Art. 78. It may happen, in consequence of some absurdity or inconsistency in the conditions of a problem, that we obtain, for a result or answer to the question, a quantity affected with the sign $-$. Such a result is called a *negative solution.*

Negative results not only indicate some absurdity or inconsistency in the conditions of a question, but also teach us how to modify the question, so as to free it from all inconsistency.

As such negative quantities frequently occur, we shall proceed to show, that, when *isolated* or standing alone, they are subject to the same rules as when connected with other quantities.

We remark, in the first place, that negative quantities are derived from attempting to subtract a greater quantity from a less. The greatest quantity that can be taken from another, is that quantity itself. Thus, 7 is the greatest number that can be subtracted from 7, and a is the greatest number that can be subtracted from a. In such a case the remainder is zero; thus, $7-7=0$, and $a-a=0$.

If it were required to subtract 9 from 7, we represent it thus, $7-9$, or $7-7-2$; this being reduced becomes -2. The sign $-$ before the 2, shows that there were 2 out of the 9 units, which could not be actually subtracted. If 7 be subtracted from 9, the remainder is the same, except that it has the sign $+$.

In like manner, if we subtract b from a, b being the greater, the remainder, $a-b$, will be negative; but if we subtract a from b, the remainder, $b-a$, will be the same as before, except that it will be positive.

Art. **79**. Suppose it were required to add $b-c$ to a. It is evident, that we are to add to a the quantity b, and subtract from the sum the quantity c, and the result is $a+b-c$.

Now, as the reasoning does not depend at all upon the value of b, the method of proceeding must be the same when $b=0$, which reduces the expression $b-c$ to $0-c$ or $-c$, and $a+b-c$ becomes $a+0-c$ or $a-c$; that is, $-c$ added to a gives $a-c$, which accords with the rule already given for adding polynomials. Hence,

Adding a negative quantity $-c$, *is equivalent to subtracting an equal positive quantity* $+c$.

Art. **80**. Since $b-b=0$, $a+b-b$ is of the same value as a, and may be regarded as the quantity a under a different form. Now, in order to subtract $+b$ from a, it is sufficient to strike it out from the expression $a+b-b$, and we have $a-b$; or if we would subtract $-b$, strike that out, and we have $a+b$; that is, $+b$ subtracted from a gives $a-b$, and $-b$ subtracted from a gives $a+b$, which accords with the rule already given for subtracting polynomials. Hence,

Subtracting a negative quantity $-b$, *is equivalent to adding an equal positive quantity* $+b$.

Art. **81**. With regard to multiplication, in Art. **30**, we have already seen that the product of $a-b$ by $c-d$ is

$$ac-bc-ad+bd.$$

Now it is manifest that the sign of any term in this product, is entirely independent of the absolute value of the letters, a, b, c, and d.

Let us suppose then, in the first place, that a and d are each equal to zero. Upon this supposition, the quantities to be multiplied together become $0-b$ and $c-0$, or $-b$ and c; and the product becomes $0 \,.\, c - bc - 0 \,.\, 0 + b \,.\, 0$, or $-bc$. Hence, $-b$ multiplied by $+c$, produces $-bc$.

Secondly, suppose b and c each equal to zero. Then the quantities to be multiplied together become a and $-d$; and the product, $ac - bc - ad + bd$, is reduced to $-ad$. Hence, $+a$ multiplied by $-d$, produces $-ad$.

Lastly, let the value of each of the letters a and c be zero. We then have to multiply $-b$ by $-d$; and the product, $ac - bc - ad + bd$, is reduced to $+bd$. Hence, $-b$ multiplied by $-d$, produces $+bd$.

From these several results we deduce the same rule for the signs in multiplication, as that given in Art. **31**.

Art. **82**. Since in division, the product of the divisor and quotient must reproduce the dividend, it follows from the preceding demonstration, that the rule for the signs in the division of isolated quantities, is the same as that given for polynomials.

We conclude then, in general, that the four fundamental operations are performed upon algebraic quantities when *isolated*, according to the same rules, in respect to the signs, as when they constitute terms of polynomials.

Art. **83**. It is manifest from what precedes, that *addition* in algebra does not always imply the idea of augmentation; for, if we add $-b$ to a, the result $a-b$ is less than a by the quantity b.

Nor does *subtraction* in algebra always imply the idea of diminution; for, if we subtract $-b$ from a, the result $a+b$ is greater than a by the quantity b.

To distinguish these results from those of addition and subtraction in arithmetic, we use the terms *algebraic sum* and *alge-*

braic difference. Thus, $a-b$ is the algebraic sum of a and $-b$; and $a+b$ is the algebraic difference between a and $-b$.

Art. 84. 1. If a rectangular field is 10 rods long and 7 rods wide, how much must be added to the length, in order that the field may contain 49 square rods?

Suppose x rods added to the length; then
$10+x=$ the length after x rods are added; hence,
$7(10+x)=49$, or $70+7x=49$.
This equation gives $x=-3$ rods.

Here the value of x being negative, indicates some absurdity in the question.

If we return to the equation $70+7x=49$, we perceive that the absurdity consists in supposing, that something must be arithmetically added to 70 to make it equal to 49.

The result, $x=-3$, shows that -3 rods must be algebraically added to the length, that is, 3 rods must be subtracted from it.

Let us then put $-x$ instead of $+x$ in the original equation, and it becomes

$$7(10-x)=49, \text{ or } 70-7x=49.$$

This gives $x=3$ rods.

The question therefore should have been as follows:

If a rectangular field is 10 rods long and 7 rods wide, how much must be subtracted from the length, that the field may contain 49 square rods?

We are conducted to this modification in the question, merely by changing the sign of x in the original equation. We see, moreover, that both equations give the same result, except with regard to the sign.

2. If a field is 9 rods long and 5 rods wide, how much must be subtracted from the length, so that the area of the field may be 65 square rods?

If $x=$ the number of rods to be subtracted, we have

$$5(9-x)=65, \text{ or } 45-5x=65.$$

This equation gives $x = -4$; hence, -4 rods are to be subtracted from the length, that is, 4 rods are to be added to it.

Indeed the equation $45 - 5x = 65$ is evidently absurd, since it supposes that something must be taken from 45 to make it equal to 65.

Let us put $+x$ instead of $-x$ in the original equation; this equation then becomes

$$5(9 + x) = 65, \text{ which gives } x = 4 \text{ rods.}$$

We see therefore that the inquiry should have been, how much must be added to the length.

3. A father whose age is 68 years, has a son aged 20; in how many years will the son be one fourth as old as his father?

Suppose $x =$ the number of years; then

$$20 + x = \frac{68 + x}{4}. \quad \text{This equation gives } x = -4.$$

Changing the sign of x in the original equation, we have

$$20 - x = \frac{68 - x}{4}, \text{ which gives } x = 4 \text{ years.}$$

The question therefore should have been; how many years ago was the son one fourth as old as his father?

4. A laborer wrought for a gentleman 7 days, having his son with him 4 days, and received 27 shillings; at another time he wrought 9 days, having his son with him 6 days, and received 33 shillings. What were the daily wages of the laborer and his son respectively?

Let $x =$ the daily wages of the man,
and $y =$ the daily wages of the boy.
Hence, $7x + 4y = 27$,
and $9x + 6y = 33$.

These equations give $x = 5$, and $y = -2$.

Changing the sign of y in the original equations, we have

$$7x - 4y = 27,$$
$$\text{and } 9x - 6y = 33.$$

These equations give $x = 5$ shillings and $y = 2$ shillings.

It appears then that the son was an expense to his father, and that the inquiry should have been: how much did the laborer receive per day for himself, and how much did he pay per day for his son?

5. What fraction is that, to the numerator of which if 1 be added, the value of the fraction will be $\frac{4}{5}$; but if 1 be added to the denominator, the value will be $\frac{5}{8}$?

Let $\dfrac{x}{y}$ be the fraction.

Then $\dfrac{x+1}{y}=\frac{4}{5}$,

and $\dfrac{x}{y+1}=\frac{5}{8}$. These equations give $x=-5$, and $y=-9$.

Here the values of x and y are both negative. Changing the signs of x and y in the original equations, we have

$\dfrac{-x+1}{-y}=\frac{4}{5}$, and $\dfrac{-x}{-y+1}=\frac{5}{8}$. But we may change the signs of the numerators and denominators of the first members without altering the value of the fractions; we then have

$\dfrac{x-1}{y}=\frac{4}{5}$, and $\dfrac{x}{y-1}=\frac{5}{8}$.

The question should, therefore, have been as follows:

What fraction is that, from the numerator of which if 1 be subtracted, the value will be $\frac{4}{5}$; but if 1 be subtracted from the denominator, the value will be $\frac{5}{8}$?

The preceding problems render it manifest, that a negative result indicates some absurdity in the conditions of the question, and show us, that the conditions are modified so as to remove the absurdity, by rendering *subtractive*, quantities which had been previously considered as *additive*, or by rendering *additive*, quantities which had previously been considered as *subtractive*.

We see moreover, that, in order to ascertain what the conditions should have been, we have only to change, in the original equations, the signs of those quantities for which we have obtained negative values, and modify the question accordingly.

Negative quantities are sometimes said to be less than zero, and, in an algebraical sense, they may be so considered. But strictly speaking, no quantity can be less than zero. When we say, for example, of a bankrupt, that he is worth $5000 less than nothing, we mean simply, that he owes $5000 more than he can pay.

Negative quantities do not, in reality, differ from positive quantities, and are merely positive quantities taken in a sense different from that first supposed.

Let the learner solve the following questions, and show how the negative results are to be interpreted.

6. What number is that, $\frac{7}{10}$ of which exceeds $\frac{5}{7}$ of it by 5?

7. A man, when he was married, was 30 years old, and his wife 20. How many years before their marriage was his age to hers as 7 to 6?

8. What fraction is that, whose value, if its denominator be diminished by 2, will be $\frac{1}{5}$, but whose value, if its numerator be diminished by 2, will be $\frac{5}{13}$?

9. Find two numbers, such that their difference shall be 20, and the difference between 6 times the greater and 3 times the less shall be 96.

10. A cistern is provided with two stopcocks, A and B, through which water flows. After the stopcock A had been open 5 minutes, and B 3 minutes, there were found 24 gallons in the cistern; but if A had been open 7 minutes and B 5, there would have been 32 gallons in the cistern. How much water flows into the cistern through each stopcock in a minute?

11. Three times A's money, twice B's, and four times C's make $13000; four times A's, three times B's, and twice C's make $25000; and six times A's, four times B's, and once C's make $40000. Required the estate of each.

SECTION XXIX.

DISCUSSION OF PROBLEMS.

Art. **85**. When a question has been resolved generally, that is, by using letters to represent the known quantities, we sometimes inquire what values the unknown quantities will assume, in consequence of particular suppositions with regard to the known quantities. The determination of these values, and the interpretation of the remarkable results which we may obtain, constitute what is called the *discussion* of the problem.

The discussion of the following problem, which is originally due to Clairaut, presents many remarkable circumstances.

1*st case.* Two couriers set out, at the same time, from two points, A and B, which are a miles asunder, and travel towards each other. The courier from A goes b miles per hour, and the courier from B, c miles per hour. At what distance from A and B will they meet?

A R B

Let R be the point of meeting. Suppose $x =$ the distance from A to R, and $y =$ the distance from B to R. Then we have

(1) $x + y = a$.

As the courier from A goes b miles per hour, he will be $\frac{x}{b}$ hours in going x miles; in like manner, the courier from B will be $\frac{y}{c}$ hours in going y miles; and since they are equal times on the road, we have

(2) $\frac{x}{b} = \frac{y}{c}$. Multiply the 2d equation by b,

$x = \frac{b\,y}{c}$; substitute this value of x in the 1st,

$\frac{by}{c}+y=a$; multiply by c,

$by+cy=ac$, or $(b+c)y=ac$; hence,

$y=\frac{ac}{b+c}=$ distance of the point of meeting from B.

Substituting the value of y in the equation $x=\frac{by}{c}$, or $x=\frac{b}{c}\times y$, we have

$$x=\frac{b}{c}\cdot\frac{ac}{b+c}=\frac{abc}{c(b+c)}=\frac{ab}{b+c}=\text{ distance of the point}$$

of meeting from A.

As the sign — does not occur in the values of x and y, these values will always be positive, whatever numbers are put instead of a, b and c. Indeed it is evident, that since the couriers travel towards each other, they must necessarily meet between A and B.

2d case. Suppose now that the couriers, setting out from the points A and B, as in the diagram below, proceed both in the same direction, and travel towards the point C, at the same rates as before. What distance will each travel before one overtakes the other?

A B R C

Suppose R the point where they come together. Let $x=$ A R, and $y=$ B R. Then,

(1) $x-y=a$, and

(2) $\frac{x}{b}=\frac{y}{c}$.

These equations being solved, give

$$x=\frac{ab}{b-c}, \text{ and } y=\frac{ac}{b-c}.$$

Here the values of x and y will not be positive, unless b is greater than c; that is, unless the courier from A travels faster than the courier from B.

Suppose $b=10$, and $c=8$; then we have

$$x=\frac{10\,a}{10-8}=\frac{10\,a}{2}=5\,a;$$

$$\text{and } y=\frac{8\,a}{10-8}=\frac{8\,a}{2}=4\,a.$$

But if we suppose that b is less than c, and that $b=8$ and $c=10$, we have

$$x=\frac{8\,a}{8-10}=\frac{8\,a}{-2}=-4\,a;\text{ and}$$

$$y=\frac{10\,a}{8-10}=\frac{10\,a}{-2}=-5\,a.$$

Here the values of x and y are both negative, and show that there is some inconsistency in the question; and indeed it is absurd to suppose, that the courier from A can overtake the courier from B, both travelling towards C, unless the former travel faster than the latter.

In order to see how the question is to be modified, let us change the signs of x and y in the original equations.

We then have

$$y-x=a, \text{ and}$$

$\dfrac{-x}{b}=\dfrac{-y}{c}$. The last equation, by a change of the signs, becomes

$$\frac{x}{b}=\frac{y}{c}.$$

It is evident that the 2d equation will remain the same as before, because it merely expresses the equality of the times.

The equation $y-x=a$ shows that y is greater than x, or that the point where they come together, is further from B than it is from A; and since this point cannot be between A and B, it must be on the other side of A with respect to B, as at R′ in the subjoined diagram.

C′ R′ A B R C

When, therefore, b is less than c, the question should be as follows:

Two couriers set out from the points A and B, a miles distant from each other, and travel towards C′; the courier from A goes b miles per hour, and the courier from B, c miles per hour; how far will each travel before the courier from B overtakes the one from A?

In this case, the equations,

$$y - x = a, \text{ and } \frac{x}{b} = \frac{y}{c}, \text{ will give}$$

$$x = \frac{ab}{c-b}, \text{ and } y = \frac{ac}{c-b}.$$

If $b = 8$ and $c = 10$, we have

$$x = \frac{8a}{10-8} = 4a, \text{ and } y = \frac{10a}{10-8} = 5a.$$

We see, in this question, that a change of sign indicates a change in direction. Numerous instances of similar indications occur in the application of algebra to geometry.

3*d case.* Resuming the formulæ,

$$x = \frac{ab}{b-c}, \text{ and } y = \frac{ac}{b-c}, \text{ let us suppose } b = c; \text{ then}$$

$b - c$ being 0, we have

$$x = \frac{ab}{0}, \text{ and } y = \frac{ac}{0}.$$

In order to interpret these results, let us go back to the original equations, $x - y = a$, and $\frac{x}{b} = \frac{y}{c}$. By putting b instead of c in the second equation, it becomes $\frac{x}{b} = \frac{y}{b}$, which gives $x = y$; and substituting x for y in the first equation, we have

$$x - x = a, \text{ or } 0 = a.$$

This result is manifestly absurd, since we have a known quantity equal to zero; and it is evident, that since the couriers travel equally fast, it is impossible one should overtake the other.

We regard therefore $\frac{m}{0}$, or any similar expression, as a *symbol of impossibility;* and when a question gives $0 = a$ (a being

any known quantity different from zero), or when the unknown quantity is found $=\frac{a}{0}$, the question is to be considered as impossible.

There is, however, another signification of such an expression as $\frac{m}{0}$, which it is important to notice.

Let us take the expressions for x and y, viz: $x=\frac{ab}{b-c}$, and $y=\frac{ac}{b-c}$.

Making $b=10$ and $c=9{\cdot}9$, we have
$$\begin{cases} x=\dfrac{10a}{10-9{\cdot}9}=\dfrac{10a}{{\cdot}1}=100a; \\ y=\dfrac{9{\cdot}9a}{10-9{\cdot}9}=\dfrac{9{\cdot}9a}{{\cdot}1}=99a. \end{cases}$$

Making $b=10$ and $c=9{\cdot}99$, we have
$$\begin{cases} x=\dfrac{10a}{10-9{\cdot}99}=\dfrac{10a}{{\cdot}01}=1000a; \\ y=\dfrac{9{\cdot}99a}{10-9{\cdot}99}=\dfrac{9{\cdot}99a}{{\cdot}01}=999a. \end{cases}$$

Making $b=10$ and $c=9{\cdot}999$, we have
$$\begin{cases} x=\dfrac{10a}{10-9{\cdot}999}=\dfrac{10a}{{\cdot}001}=10000a; \\ y=\dfrac{9{\cdot}999a}{10-9{\cdot}999}=\dfrac{9{\cdot}999a}{{\cdot}001}=9999a. \end{cases}$$

We here perceive that the value of the fraction increases in proportion as the denominator is diminished; if then the denominator be less than any assignable quantity, that is 0, the value of the fraction will be greater than any assignable quantity, or infinitely great. Hence mathematicians consider a fraction, whose numerator is a definite quantity, and whose denominator is zero, as a *symbol of infinity*. Thus, $\frac{6}{0}, \frac{a}{0}, \frac{m+c}{0}$, are *symbols of infinity*.

Remark. In the problems of geometry, solved by the aid of algebra, there are many instances, in which an infinite quantity, instead of denoting an absurdity, is the true result sought.

If a definite quantity be divided by an infinite or impossible quantity, the quotient will be zero. Thus a divided by $\frac{m}{0}$ gives $\frac{a \cdot 0}{m} = \frac{0}{m} = 0$.

4th case. Suppose now, that, in the formulæ for x and y, $b = c$ and $a = 0$.

The distance between the points A and B being nothing, these points must be coincident, as in the following diagram,

A

B

and the formulæ for x and y become

$x = \frac{0 \cdot b}{0} = \frac{0}{0}$, and $y = \frac{0 \cdot c}{0} = \frac{0}{0}$; or $x \cdot 0 = 0$, that is, $0 = 0$, and $y \cdot 0 = 0$, that is, $0 = 0$.

Now, as the couriers set out from the same point, and travel equally fast and in the same direction, they cannot be said to come together at any particular point, since they are constantly together throughout the whole route.

But in order to see the general import of the expression $\frac{0}{0}$, let us return to the original equations, $x - y = a$, and $\frac{x}{b} = \frac{y}{c}$.

Putting 0 instead of a, and b instead of c, we have $x - y = 0$, and $\frac{x}{b} = \frac{y}{b}$.

The first equation gives $x = y$, and this value of x being substituted in the second, gives $\frac{y}{b} = \frac{y}{b}$.

This last equation, in which the two members are precisely alike, is called an *identical equation*, and is verified by putting any quantity whatever instead of y. The value of y therefore cannot be determined from this equation.

Moreover the equation $\frac{x}{b} = \frac{y}{b}$, gives $x = y$, and therefore expresses nothing more than the first.

Hence the expression $\frac{0}{0}$ is a *symbol of an indeterminate quantity;* and when a problem results in giving $0 = 0$, or when the unknown quantity is found $= \frac{0}{0}$, the question is to be regarded as indeterminate.

There are however some precautions to be used, before we decide that a quantity is indeterminate.

Thus, $\dfrac{a^3 - b^3}{a - b}$, when $a = b$, becomes $\frac{0}{0}$; but the numerator and denominator both being divisible by $a - b$, if the fraction is reduced, it becomes $a^2 + a b + b^2$ or $3 a^2$, which is a determinate quantity.

When, therefore, we arrive at a result $= \frac{0}{0}$, before we pronounce it indeterminate, we must see whether the fraction which represents this result, has not a factor common to its numerator and denominator, which being struck out, will render the quantity definite.

Let the learner solve the following problems and interpret the results.

1. A boy being asked how much money he had, replied, that $\frac{1}{3}$ and $\frac{5}{12}$ of his money, increased by 40 cents, would be equal to $\frac{3}{4}$ of his money, increased by 49 cents. How many cents had he?

2. Four men, A, B, C and D, talking of their ages, it was found that B was 10 years older than A and 10 years younger than C, and that D was 34 years younger than A; moreover, that $\frac{1}{2}$ of B's age, $\frac{2}{3}$ of C's, and $\frac{5}{6}$ of D's would be equal to twice A's diminished by 10 years. Required the age of each.

3. How many ducks have you killed to-day, said a farmer to a sportsman; the latter replied, one half of the number I have killed to-day, exceeds $\frac{1}{3}$ of what I killed yesterday by 5; and the number I killed yesterday, is 5 less than once and a half the number I have killed to-day. Required the number he killed each day.

SECTION XXX.

EXTRACTION OF THE SECOND ROOTS OF NUMBERS.

Art. **86**. What number is that, which, being multiplied by 7 times itself, gives a product equal to 448?

Let x represent the number; then $x \,.\, 7x$ or $7x^2 = 448$.

This is called an equation of the *second degree*, because it contains the second power of the unknown quantity. Such an equation is also sometimes called a *pure quadratic equation.*

In order to solve this equation, we first divide by 7, and have

$$x^2 = 64, \text{ or } x \,.\, x = 64.$$

Hence, x must be a number, which multiplied by itself, will give 64; and we know that $8 \,.\, 8 = 64$; therefore $x = 8$.

The first power of a quantity, in reference to the second, is called the *root*, and finding the first power when the second is given, is called *extracting the second root.* The second root of a quantity then, is such as being multiplied by itself, will produce the given quantity.

The second powers of the first nine figures, are as follows.

$$\begin{cases} 1,\ 2,\ 3,\ \ 4,\ \ 5,\ \ 6,\ \ 7,\ \ 8,\ \ 9. & \text{Roots.} \\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81. & \text{Powers.} \end{cases}$$

We perceive from this table, that when a number contains only one figure, its second power cannot contain more than two figures. The least number containing two figures is 10, the second power of which is 100, consisting of three figures.

In order to find a rule for extracting the roots of numbers containing more than two figures, let us see how a second power is formed from its root.

The second power of $a + b$ is $a^2 + 2ab + b^2$. Suppose $a = 20$ or 2 tens, and $b = 5$; then $a^2 = 400$, $2ab = 2 \,.\, 20 \,.\, 5 = 200$, and $b^2 = 25$; hence $a^2 + 2ab + b^2 = 400 + 200 + 25 = 625$.

When, therefore, a number contains tens and units, its second power will contain the second power of the tens, plus twice the product of the tens by the units, plus the second power of the units.

Now let us reverse the process, and see by what means the root could be found from the power.

Operation.

$$\begin{array}{l} 6'25(\underline{25.}\ \text{ Root.} \\ \underline{4} \\ 22'5(\underline{4.}\ \text{ Divisor.} \end{array} \qquad 25 \,.\, 25 = 625.$$

Since the second power of the tens of the root can contain no significant figure below hundreds, it must be found in the 6, that is, 6 (hundreds); we therefore separate the last two figures from the 6 by an accent, placed over the top. Now, because the second power of 3 (tens) is 9 (hundreds), and that of 2 (tens) is 4 (hundreds), the latter is the greatest second power of tens contained in 6 (hundreds), and the root is 2 (tens). We place 2 at the right of the proposed number, separating it by a line, as is done with the quotient in division, and subtract the second power, 4 (hundreds) or a^2, from 6 (hundreds).

To the right of the remainder 2, we bring down the two figures cut off, and have 225. This number corresponds to $2\,a\,b + b^2$; that is, it contains twice the product of the tens of the root by the units, plus the second power of the units. If it contained $2\,a\,b$ only, or twice the product of the tens by the units, we should obtain the units exactly by dividing by $2\,a$, or twice the tens. As it is, if we divide by twice the tens, disregarding the remainder, we shall obtain the units exactly, or a number a little too great.

But since twice the tens multiplied by the units, cannot have any significant figure below tens, if we take 4 merely as the divisor, we must reject the right hand figure, 5, of the dividend. Or, in other words, since the divisor is ten times too small, if we make the dividend ten times too small, the quotient will not be affected by this change. The divisor 4 is contained in 22 five

times. Putting 5 at the right of the 2 in the root, we have 25, which raised to the second power gives 625. Hence 25 is the root sought.

We shall now explain how the correctness of any figure in the root may be ascertained, without raising the whole to the second power.

Let it be required to extract the root of 1521.

Operation.

15'21(39. Root.
9
62'1(69. Divisor.
621
0.

Reasoning as before, we find the greatest second power of tens contained in 15 (hundreds), to be 9 (hundreds), the root of which is 3 (tens); putting 3 as the first figure of the root, and subtracting its second power from 15, we bring down the next two figures, and have for a dividend 621. This corresponds to $2ab + b^2$, which is the same as $b(2a + b)$. Dividing 62 by 6, twice the tens, we have for a quotient 10; but as the unit figure cannot exceed 9, we put 9 in the root at the right of the 3, and we have 39 for the entire root.

In order to determine whether 9 is the proper unit figure of the root, we observe that the divisor 6 (tens) corresponds to $2a$, and 9 is the figure which we have found for b; hence, $60 + 9$ or 69 corresponds to $2a + b$; therefore we place 9 at the right of the divisor and multiply 69 by 9; the product 621 answers to $b(2a + b)$; this subtracted from the dividend leaves nothing. Therefore the true root is 39.

Let the learner extract the roots of the following numbers by the process last explained.

1. 784.
2. 2809.
3. 6084.
4. 841.
5. 1296.
6. 8649.
7. What is the second root of 127449?

The second powers of 10, 100, 1000 are respectively 100, 10000, 1000000; hence the second power of any whole number between 10 and 100, that is, consisting of two figures, will be between 100 and 10000, that is, it will contain three or four figures; also, the second power of a number consisting of three figures, will contain five or six figures. We can, therefore, ascertain the number of figures in the root of any proposed number, by beginning at the right, and separating it into parts or periods of two figures each. The left hand period may consist of one or two figures. There will be as many figures in the root, as there are periods in the power.

Separating 127449 into periods, we see that the root must contain three figures, or hundreds, tens, and units.

Let a represent the hundreds of the root, b the tens, and c the units; then $a+b+c$, regard being paid to the rank of the figures, will represent the root.

The second power of $a+b+c$ is $a^2+2ab+b^2+2ac+2bc+c^2$, or $a^2+2ab+b^2+2(a+b)c+c^2$. Hence, the second power contains the second power of the hundreds, plus twice the product of the hundreds by the tens, plus the second power of the tens, plus twice the sum of the hundreds and tens multiplied by the units, plus the second power of the units. We proceed now to extract the root.

Operation.

```
12'74'49(357.  Root.
 9          = a².
 37'4(65    = 2a + b.
 325        = (2a + b) b.
  494'9(707 = 2(a + b) + c.
  4949      = [2(a + b) + c] c.
     0.
```

We first seek the second power of the hundreds of the root, which must be found in the 12, (120000); the greatest second power in this part is 9, (90000), the root of which is 3, (300).

Putting 3 as the first figure of the root, and subtracting its second power from 12, we bring down the next period at the right of the remainder. We now consider 374 as a dividend.

This dividend contains $2ab + b^2$, or twice the product of the hundreds by the tens, plus the second power of the tens, together with the hundreds arising from multiplying twice the sum of the hundreds and tens by the units.

It is now our object to find b, or the tens of the root; and for this purpose, we divide by $2a$ or twice the hundreds. But as the product of twice the hundreds by the tens, can have no significant figure below the fourth place, in dividing we reject the right hand figure of the dividend, separating it by an accent.

We double the hundreds, and obtain 6 for a divisor, which is contained in 37 six times.

But if we put 6 at the right of the divisor and multiply 66 by 6, we obtain a product greater than 374. We next try 5, which we place in the root and also at the right of the divisor, and we have 65 corresponding to $2a + b$; this multiplied by 5 gives 325, corresponding to $(2a + b)b$.

We now subtract 325 from the dividend, to the remainder annex the figures of the last period, and obtain for a new dividend 4949.

This dividend contains $2(a + b)c + c^2 = [2(a + b) + c]c$, or twice the sum of the hundreds and tens multiplied by the units, plus the second power of the units. To obtain the units, therefore, we must divide by twice the hundreds and tens already found.

But as hundreds and tens multiplied by units, can have no significant figure below tens, we reject the right hand figure of the dividend, separating it by an accent. Double the hundreds and tens makes 70, (700), $= 2(a + b)$, which is contained in 494, (4940), seven times.

We then put 7, which corresponds to c, in the root, and also at the right of the divisor, and we have $707 = 2(a + b) + c$; this multiplied by 7 gives $4949 = [2(a + b) + c]c$, which sub-

tracted from the dividend, leaves no remainder. Therefore 357 is the root sought.

If the root contains more than three figures, representing the figure of the highest rank by a, the next by b, &c., we have for the second power $a^2 + 2ab + b^2 + 2(a+b)c + c^2 + 2(a+b+c)d + d^2 + 2(a+b+c+d)e + e^2$, &c., or $a^2 + (2a+b)b + [2(a+b)+c]c + [2(a+b+c)+d]d + [2(a+b+c+d)+e]e$ &c.; the first form of which shows, that, after the first figure has been found, each of the successive figures is obtained by dividing by twice the whole root already found; and the second form shows, that, in each case, the quotient is to be placed at the right of the divisor, and that the divisor thus increased, is to be multiplied by the quotient.

Moreover, from a consideration of the rank of the figures, it is plain, that twice the root already found, multiplied by the next lower figure, can produce no significant figure below the second from the right in each dividend.

8. What is the second root of 1024832169?

Operation.

```
10'24'83'21'69(32013.  Root.
 9
 12'4(62
 124
    832'1(6401
    6401
    19206'9(64023
    192069
         0.
```

Operating in this question as in the preceding ones, we find that the second divisor 64, is not contained in the dividend 83, the right hand figure being rejected, which shows that there are no hundreds in the root sought; in this case, we place a zero in the root, also at the right of the divisor, and bring down the next two figures to form a dividend.

We may observe, that, if the last figure of the preceding divisor be doubled, the root, so far as it is ascertained, will be doubled; for that divisor contains twice this root, with the exception of the figure last found.

Art. 87. From the preceding analysis we derive the following

RULE FOR EXTRACTING THE SECOND ROOTS OF NUMBERS.

1. *Begin at the right, and, by means of accents, separate the number into periods of two figures each. The left hand period may contain one or two figures.*

2. *Find the greatest second power in the left hand period, place its root at the right of the proposed number, separating it by a line, and subtract the second power from the left hand period.*

3. *To the right of the remainder bring down the next period to form a dividend. Double the root already found for a divisor. Seek how many times the divisor is contained in the dividend, rejecting the right hand figure. Place the quotient in the root, at the right of the figure previously found, and also at the right of the divisor. Multiply the divisor thus increased by the last figure of the root, and subtract the product from the whole dividend.*

4. *Bring down to the right of the remainder the next period, to form a new dividend. Double the root already found for a divisor, and proceed as before to find the third figure of the root.*

Repeat this process until all the periods have been brought down.

Remark. *If the dividend will not contain the divisor, the right hand figure of the former being rejected, place a zero in the root, also at the right of the divisor, and bring down the next period.*

Extract the roots of the following numbers.

1. 1369.
2. 2401.
3. 361.
4. 123201.
5. 502681.
6. 11881.
7. 36100.
8. 1100401.
9. 1432809.
10. 151905625.
11. 901260441.
12. 2530995481.

Art. 88. There are comparatively but few numbers which are exact second powers; and the roots of such as are not perfect powers, cannot be obtained exactly either in whole numbers or fractions. For example, the root of 42 is between 6 and 7; but no number can be found, which, multiplied by itself, will produce exactly 42. We shall however see hereafter, that the root of any positive number may be approximated to any degree of exactness.

Since the roots of numbers, which are not perfect powers, cannot be obtained exactly, either in whole or fractional numbers, they are said to be *irrational*, or *incommensurable;* that is, these roots and unity have no common divisor. Roots of other degrees, besides the second, are also called *irrational*, when they cannot be exactly obtained.

The second root of a quantity, whether that root can be found exactly or not, is *indicated* either by the exponent $\frac{1}{2}$, or by this character $\sqrt{}$, called the *radical sign*. Thus, $(25)^{\frac{1}{2}}$ or $\sqrt{25} = 5$; and $(3)^{\frac{1}{2}}$ or $\sqrt{3}$ means the second root of 3.

But the second root of a negative quantity cannot be obtained, even by approximation, since there is no number, which, multiplied by itself, can give a negative quantity. The second roots, therefore, of negative quantities are called *imaginary*, in opposition to those of positive quantities, which are *real*, although they cannot be exactly obtained. Thus, $(-16)^{\frac{1}{2}}$ or $\sqrt{-16}$ is imaginary. These imaginary quantities, except in some of the higher branches of analysis, indicate absolute absurdity in the questions from which they arise.

SECTION XXXI.

SECOND ROOTS OF FRACTIONS—AND THE EXTRACTION OF SECOND ROOTS BY APPROXIMATION.

Art. **89**. The second power of a fraction is found by raising both numerator and denominator to the second power; for this is equivalent to multiplying the fraction by itself. Thus,

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}.$$

Hence, the second root of a fraction is found by extracting the root of the numerator and that of the denominator. Thus, the root of $\frac{16}{49}$ is $\frac{4}{7}$, and that of $\frac{a^2}{b^2}$ is $\frac{a}{b}$.

Let the roots of the following fractions be found.

1. $\frac{16}{25}$.
2. $\frac{49}{64}$.
3. $\frac{81}{121}$.
4. $\frac{169}{289}$.
5. $\frac{625}{1089}$.
6. $\frac{6241}{12544}$.

Art. **90**. If, however, either the numerator or denominator is not a perfect second power, the root of the fraction can be obtained by approximation only. Thus, the root of $\frac{15}{49}$ is between $\frac{3}{7}$ and $\frac{4}{7}$. It is nearer to $\frac{4}{7}$.

The denominator of a fraction, however, may always be rendered a perfect second power, by multiplying both numerator and denominator by the denominator, which does not alter the value of the fraction. For example, $\frac{3}{7} = \frac{21}{49}$, the approximate root of which is $\frac{5}{7}$—. By this mode, the root has the same denominator as the given fraction.

Remark. The sign $+$, placed after an approximate root, signifies that it is less, and the sign —, that it is greater than the true one.

When a greater degree of exactness is requisite, we may, after having multiplied both terms of the fraction by its denominator, multiply both terms of the result by any second power.

Thus, $\frac{3}{7} = \frac{21}{49} = \frac{144 \,.\, 21}{144 \,.\, 49}$ or $\frac{3024}{7056}$, the approximate root of which is $\frac{55}{84}$ —.

Let the learner find the roots of the following fractions, in the denomination of their respective denominators.

1. $\frac{7}{8}$.	4. $\frac{3}{22}$.
2. $\frac{5}{12}$.	5. $\frac{7}{15}$.
3. $\frac{3}{11}$.	6. $\frac{21}{25}$.

Art. **91**. The root of a whole number may be approximated in the same way, by converting it into a fraction, having a second power for its denominator. If, for example, we would find the root of 5, exact to 12ths, we change 5 to the fraction $\frac{720}{144}$, the approximate root of which is $\frac{27}{12}$ —.

But it is most convenient to change the number into a fraction, having the second power of 10, 100, or 1000, &c., for a denominator; that is, convert the number into 100ths, 10000ths, or 1000000ths, &c., and the root will be found in decimals.

Thus, $5 = \frac{500}{100} = \frac{50000}{10000} = \frac{5000000}{1000000}$; that is, $5 = 5{\cdot}00 = 5{\cdot}0000 = 5{\cdot}000000$; the approximate root of the first is $\frac{22}{10} + = 2{\cdot}2 +$, of the second $\frac{223}{100} + = 2{\cdot}23 +$, and of the third $\frac{2236}{1000} + = 2{\cdot}236 +$.

In the example just given, we perceive that twice as many zeros are annexed to the number, as we wish to have decimals in the root. Indeed, it is plain, that there must be half as many decimals in the root as there are in the power, because the second power of 10ths produces 100ths, the second power of 100ths produces 10000ths, &c.

Moreover, we need not add all the zeros at once, but may annex two to each remainder, in the same manner as we bring down the figures of successive periods.

As an example, let us extract the second root of 3.

Operation.

```
3·(1·732+
1
20'0(27
189
 110'0(343
 1029
   710'0(3462
   6924
    176.
```

The operation might be continued to any extent.

The process will be the same for any number containing decimals; and any fraction may be converted into decimals, and the root may be extracted in the same way, care being taken to make the number of decimals even.

It is best, when the number contains decimals, to begin at the decimal point, and separate the decimals into periods by proceeding towards the right, and the whole numbers by proceeding towards the left.

Art. **92**. In approximating a second root, we may sometimes be in doubt, whether the last figure found is so great as it should be. This may be determined in the following manner.

The second power of a is a^2, and that of $a+1$ is a^2+2a+1. Now the root of a^2+2a+1 is $a+1$; but if we should call a its root, and subtract the second power of a, there would remain $2a+1$. Hence, when the root admits of being increased by 1, the remainder will contain at least twice the root already found plus 1, the local value of the figures being disregarded.

Thus, if, in the last example, we had obtained 1·731 instead of 1·732 for the root, the remainder would have been 3639, which exceeds twice 1731 by more than 1.

Let the roots of the following numbers be found in decimals, carried to four decimal places.

1. 2.
2. 27.
3. 33·75.
4. 147·307.
5. $34\frac{3}{7}$.
6. $325\frac{15}{16}$.
7. $\frac{8}{11}$.
8. $\frac{77}{112}$.
9. $\frac{3}{4}$.
10. $\frac{1}{645}$.
11. $\frac{3}{1957}$.
12. $4\frac{37}{915}$.

SECTION XXXII.

QUESTIONS PRODUCING PURE EQUATIONS OF THE SECOND DEGREE.

Art. **93**. A *pure equation of the second degree*, or a *pure quadratic equation*, is one which contains the second power, but no other power, of the unknown quantity.

1. A's age is to B's as 7 to 9, and the sum of the second powers of their ages is 1170 years. Required their ages.

2. Two couriers set out, at the same time, from two places 220 miles asunder, and traveled towards each other till they met; when it was found that the first had traveled only $\frac{5}{6}$ as fast as the second, and that the number of hours they had been on the road, was equal to the number of miles the first traveled per hour. Required the rate per hour and the distance each traveled in the whole.

3. A gentleman has two square rooms, the sides of which are as 5 to 6; and he finds that it takes 11 square yards more of carpeting to cover the floor of the larger, than it does to cover that of the smaller. Required the length of one side of each room.

4. A farmer had an orchard, in which the number of trees in each row was to the number of rows as 6 to 5; and the number of bushels of apples, gathered from each tree, was to the number of rows as 4 to 5; moreover, the number of bushels in the whole was equal to 80 times the number of trees in one row.

How many rows were there, how many trees in each row, and how many bushels of apples were gathered?

5. A gentleman has a rectangular piece of land 50 rods long and 18 wide, which he wishes to exchange for another of the same area and in a square form. What must be the length of one side of the square?

6. A man wishes to make a cistern containing 800 gallons, the bottom of which shall be a square, and the height 6 feet. Required the length of one side of the bottom.

Note. A gallon wine measure is 231 cubic inches.

7. An acre contains 160 square rods. What is the length of one side of a square containing an acre of land?

8. What would be the length of one side of a square containing 12 acres?

9. What number is that, to which if 10 be added, and from which if 10 be subtracted, the product of the sum and difference will be 156?

10. The product of two numbers is 900, and the quotient of the greater divided by the less is 4. What are those numbers?

11. There is a house, whose breadth is to its length as 5 to 6, and whose height, exclusive of the roof, is to its breadth as 4 to 5. Required the dimensions of the house, supposing that it takes 2200 square feet of boards to cover the four sides.

12. A merchant bought two pieces of cloth of equal length; the one cost 5 shillings a yard more, and the other 5 shillings a yard less, than the number of yards in each piece. The price of the whole being £136 18s., how many yards were there in each piece, and what was the price of each per yard?

13. A company at a tavern found that their whole bill was $45, and that each had to pay 5 times as many cents as there were persons in the company. How many persons were there, and how much had each to pay?

14. There are two numbers, the sum of whose second powers is 5274, and the difference of these powers is 1224. Required the numbers.

15. A man lent a certain sum of money at 6 per cent. a year, and found that if he multiplied the principal by the number representing the interest for 8 months, the product would be $900. Required the principal.

SECTION XXXIII.

AFFECTED EQUATIONS OF THE SECOND DEGREE.

Art. **94.** The equations of the second degree, which we have hitherto considered, involved the second power only of the unknown quantity. But, in its most general sense, an equation of the second degree, with one unknown quantity, is composed of three kinds of terms, viz: one kind containing the *second power* of the unknown quantity; another containing the *first power* of the unknown quantity; and a third composed wholly of *known quantities.*

Such are called *affected equations of the second degree,* or *affected quadratic equations.*

1. There is a rectangular field whose length exceeds its breadth by 8 rods, and whose area is 180 square rods. Required the length and breadth.

Let $x =$ the breadth in rods;
then $x + 8 =$ the length.
Hence, $x^2 + 8x = 180$.

If we compare the first member of this equation with the second power of $x + a$, which is $x^2 + 2ax + a^2$, we see that it contains two terms which correspond respectively to the first two terms of this second power, viz:

$$x^2 = x^2,$$
$$2ax = 8x. \quad \text{Hence,}$$
$$2a = 8,$$
$$a = 4,$$
$$a^2 = 16.$$

Now since 16 corresponds to a^2, if we add 16 to both members of the equation, $x^2+8x=180$, the first member becomes a perfect second power corresponding to $x^2+2ax+a^2$, and we have

$$x^2+8x+16=180+16=196.$$

We now take the root of each member. The root of the first member is $x+4$, for $(x+4)(x+4)=x^2+8x+16$; and that of the second member is 14. We have therefore,

$$x+4=\pm 14.$$

Every positive quantity has two second roots, one positive and the other negative; for the second power of $-a$, as well as that of $+a$, is $+a^2$. Therefore, since in an equation such as $x+4=\pm 14$, the value of x is not determined until the known quantity is transposed, and it may happen that the negative as well as the positive root will answer the conditions of the question, we place the double sign $\pm$ before the second member. This sign is read *plus or minus*.

In the above equation, transposing 4, we have

$x=-4\pm 14$. Calling 14 plus,
$x=10$ rods, the breadth; and
$x+8=18$ rods, the length. Calling 14 minus,
$x=-18$, and $x+8=-10$.

The first value only of x answers the conditions of the question. The second value will however satisfy the equation; for, $(-18)^2+8(-18)=324-144=180$.

In order to interpret the negative value, we substitute $-x$ for $+x$ in the original equation, and we have $x^2-8x=180$, or $x(x-8)=180$. This shows that x now represents the longer side instead of the shorter. The solution of the equation, $x^2-8x=180$, will be similar to that of the following question.

2. Twenty times a certain number exceeds its second power by 75. What is that number?

Let $x=$ the number.
Then, $x^2+75=20x$. Transposing,
$$x^2-20x=-75.$$

In this equation the term containing the first power of x being negative, in order to render the first member a perfect second power, we compare it with the second power of $x-a$, which is $x^2-2ax+a^2$, and we have

$$\begin{aligned} x^2 &= x^2, \\ -2ax &= -20x. \quad \text{Hence,} \\ -2a &= -20, \\ -a &= -10, \\ a^2 &= 100. \end{aligned}$$

Adding 100 to each member, we have

$$x^2-20x+100=-75+100=25.$$

We now take the root of each member. The root of the first is $x-10$; for, $(x-10)(x-10)=x^2-20x+100$, and that of the second is ± 5. Therefore,

$$\begin{aligned} x-10 &= \pm 5. \quad \text{Transposing,} \\ x &= 10 \pm 5; \text{ hence,} \\ x &= 15, \text{ or } x=5. \end{aligned}$$

Both values of x are positive, and, therefore, both answer the conditions of the question. Indeed,

$$15 \cdot 15+75=20 \cdot 15; \text{ also, } 5 \cdot 5+75=20 \cdot 5.$$

Hence we see the propriety of giving the double sign to the root of the second member.

Art. **95.** Any affected equation of the second degree may be reduced to the form of $x^2+px=q$, p and q being any known quantities, positive or negative.

It is manifest that an equation may be reduced to this form in the following manner. 1. *Clear the equation of fractions if necessary; transpose all the terms containing x^2 and x into the first member, and the known terms into the second; reduce the terms which contain x^2 into one term, and those which contain x into another; also, reduce the known quantities in the second member to one term.* 2. *If the term containing x^2 is not positive, make it so by changing all the signs.* 3. *If the coëfficient of x^2 is not* 1, *divide all the terms by that coëfficient.*

1. A draper bought a quantity of cloth for £27. If he had bought 3 yards less for the same sum, it would have cost him 15 shillings a yard more. How many yards did he buy?

Let $x =$ the number of yards.

Then $\frac{27}{x} =$ the price per yard in pounds, and

$\frac{27}{x-3} =$ the price per yard, if he had bought 3 yards less. Hence, $\frac{27}{x-3} = \frac{27}{x} + \frac{3}{4}$. Clear the equation of fractions,

$$108x = 108x - 324 + 3x^2 - 9x;$$

transpose, reduce, and change the signs,

$$3x^2 - 9x = 324; \text{ divide by 3,}$$

$$x^2 - 3x = 108.$$

Here $p = -3$, and $q = 108$.

Comparing the first member with $x^2 - 2ax + a^2$, we have

$$\begin{aligned} x^2 &= x^2, \\ -2ax &= -3x, \\ -2a &= -3, \\ -a &= -\tfrac{3}{2}, \\ a^2 &= \tfrac{9}{4}. \end{aligned}$$

Adding $\frac{9}{4}$ to each member of $x^2 - 3x = 108$, we have

$$x^2 - 3x + \tfrac{9}{4} = 108 + \tfrac{9}{4} = \tfrac{432}{4} + \tfrac{9}{4} = \tfrac{441}{4}$$

We now take the root of each member. The root of the first is $x - \frac{3}{2}$, because $(x - \frac{3}{2})(x - \frac{3}{2}) = x^2 - 3x + \frac{9}{4}$; and that of the second is $\pm \frac{21}{2}$.

Hence, $x - \frac{3}{2} = \pm \frac{21}{2}$. Transposing $-\frac{3}{2}$,

$$x = \tfrac{3}{2} \pm \tfrac{21}{2} = \tfrac{24}{2} = 12; \text{ or } x = -\tfrac{18}{2} = -9.$$

The first value only of x answers the conditions of the question.

In order to interpret the second value of x, viz: $x = -9$, we substitute $-x$ for $+x$ in the original equation, which then becomes $\frac{27}{-x-3} = \frac{27}{-x} + \frac{3}{4}$, or $-\frac{27}{x+3} = -\frac{27}{x} + \frac{3}{4}$, since

when the denominator is negative, it is the same as if the denominator were positive and the whole fraction were preceded by the sign —.

This last equation becomes by transposition,

$\frac{27}{x} = \frac{27}{x+3} + \frac{3}{4}$, which answers to the following question.

A draper bought a quantity of cloth for £27. If he had bought 3 yards *more* for the same sum, it would have cost him 15 shillings per yard *less*.

Let the learner solve the question as now stated.

2. Since p may always represent the coëfficient of the first power of the unknown quantity, and q the known term, let us solve the general equation $x^2 + p\,x = q$, by comparing the first member with $x^2 + 2\,a\,x + a^2$. We have

$$x^2 = x^2,$$
$$2\,a\,x = p\,x,$$
$$2\,a = p,$$
$$a = \frac{p}{2},$$
$$a^2 = \frac{p^2}{4}.$$

Adding $\frac{p^2}{4}$ to each member, we have

$$x^2 + p\,x + \frac{p^2}{4} = q + \frac{p^2}{4}.$$

We now take the root of each member. That of the first member is $x + \frac{p}{2}$, since $\left(x+\frac{p}{2}\right)\left(x+\frac{p}{2}\right) = x^2 + p\,x + \frac{p^2}{4}$. The root of the second member can only be indicated, until definite values are assigned to p and q. We have then,

$$x + \frac{p}{2} = \pm\left(q + \frac{p^2}{4}\right)^{\frac{1}{2}};\ \text{hence,}$$

$$x = -\frac{p}{2} \pm \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}.$$

Remark. The second root of a quantity is properly expressed by the exponent $\frac{1}{2}$. For the second root of a quantity multiplied by itself, must produce that quantity. Now, according to the rule for exponents, Art. **19**, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1$ or a; therefore the second root of a is $a^{\frac{1}{2}}$. In like manner, the second root of $q+\frac{p^2}{4}$ is expressed thus, $\pm\left(q+\frac{p^2}{4}\right)^{\frac{1}{2}}$.

Art. **96**. From the solution of the preceding general equation, we deduce the following

RULE FOR SOLVING AFFECTED EQUATIONS OF THE SECOND DEGREE.

1. *Reduce the equation to the form of $x^2+px=q$.*
2. *Make the first member a perfect second power, by adding to both members the second power of half the coëfficient of x, (or of the first power of the unknown quantity).*
3. *Extract the root of each member. The root of the first member will consist of two terms, the first of which is x, or the unknown quantity, and the second is half the coëfficient previously found, and the root of the second member must have the double sign $\pm$.*
4. *Transpose the known term from the first member to the second and reduce, and the value of x will be found.*

Since p and q may be either positive or negative, it is evident that the general equation admits of four forms, differing only with regard to the signs of these known quantities, viz:

(1) $x^2+px=q$; whence, $x=-\frac{p}{2}\pm\left(q+\frac{p^2}{4}\right)^{\frac{1}{2}}$.

(2) $x^2-px=q$; whence, $x=+\frac{p}{2}\pm\left(q+\frac{p^2}{4}\right)^{\frac{1}{2}}$.

(3) $x^2+px=-q$; whence, $x=-\frac{p}{2}\pm\left(\frac{p^2}{4}-q\right)^{\frac{1}{2}}$.

(4) $x^2-px=-q$; whence, $x=+\frac{p}{2}\pm\left(\frac{p^2}{4}-q\right)^{\frac{1}{2}}$.

Art. **97**. These formulæ for x enable us to solve an affected equation immediately according to the following

RULE.

When the equation is reduced to the form of $x^2 + p x = q$, the unknown quantity is equal to half the coëfficient of its first power, taken with the contrary sign, plus or minus the root of the algebraic sum, obtained by adding the second power of half that coëfficient to the known term.

1. Ten times a certain number exceeds its second power by 9. What is that number?

Suppose $x =$ the number.

Then, $10x = x^2 + 9$. Transpose and change the signs,

$$x^2 - 10x = -9.$$ Hence, by the last rule,

$$x = 5 \pm (25 - 9)^{\frac{1}{2}} = 5 \pm 4 = 9;\ \text{or} = 1.$$

2. Divide the number 20 into two parts, such that their product shall be 120.

Let $x =$ one part; then will $20 - x =$ the other; and

$$20x - x^2 = 120;\ \text{or}\ x^2 - 20x = -120.$$

Solving this equation, we have

$$x = 10 \pm (100 - 120)^{\frac{1}{2}} = 10 \pm (-20)^{\frac{1}{2}}.$$

As the second root of a negative quantity is *imaginary*, this problem is impossible.

Indeed, to generalize this question, let us suppose it is required to divide the number p into two parts, such that their product shall be q.

Representing the two parts by x and $p - x$, we have

$p x - x^2 = q$; or $x^2 - p x = -q$. This equation gives

$$x = \frac{p}{2} \pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}.$$

The quantity $\pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}$ becomes imaginary whenever q is greater than $\frac{p^2}{4}$. The greatest value that can be given to q, without rendering the problem impossible, is $\frac{p^2}{4}$. Then $\left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}$ becomes zero, and x, as well as $p - x$, is equal to $\frac{p}{2}$.

The product of the two parts is then equal to $\frac{p^2}{4}$. Hence, the greatest product that can be produced by multiplying the two parts of a number together, is the second power of half that number.

3. There is a square garden, the number of square rods in which exceeds the number of rods round it by 165. Required the length of one side.

4. A man built a certain piece of wall for $27; and he found that the number of dollars he received per rod, was 6 less than the number of rods in the length of the wall. Required the number of rods and the price per rod.

5. The difference of two numbers is 5, and the sum of their second powers is 15325. What are those numbers?

6. A farmer bought, at $1 per square rod, a rectangular field, the length of which was to the breadth as 5 to 3. After having built a wall round it, which cost $2 a rod, he found that the purchase money, together with the expense of fencing, amounted to $6640. Required the dimensions of the field.

7. A drover bought a certain number of sheep for $50, and a number of calves, greater than that of his sheep by 3, for $52. Moreover, the price of a sheep and a calf together was $9. Required the number of each kind.

8. A man having traveled 160 miles, found that if he had traveled one mile more per hour, he would have been 8 hours less upon the road. Required his rate of traveling and the number of hours he was upon the road.

9. A jockey sold a horse for $150, and gained half as much per cent. as the horse cost him. How much did the horse cost him?

10. A man bought a square piece of land for $2 per square foot. After having sold from it, at the rate of $3½ per foot, a rectangular piece, the length of which was equal to one side of the square and the breadth 30 feet, he found that what remained had cost him only $4400. Required the length of one side of the square.

11. A grocer bought 60 lbs. of coffee and 80 lbs. of tea for \$46; but he found that \$1 would buy 8 lbs. more of coffee than it would of tea. Required the price of the tea and coffee per pound.

12. There is a public square whose side is 80 rods long, surrounded by a walk of uniform breadth, which contains 1344 square rods. Required the breadth of the walk.

13. What number is that to which if its second root be added, the sum will be 240?

14. A father left an estate of \$30000 to be divided equally among his sons; but one of these dying immediately after his father, the estate was divided among those remaining, each of whom received \$1500 more than he would have received, if all had been living. How many sons did the father leave?

15. A poulterer had a certain number of fowls, each of which produced, during the year, three times as many chickens as there were fowls; and, at the end of the year, he found that his whole stock, young and old, was 444. How many fowls had he at first?

16. Two men, A and B, traded together. A put in a certain sum for 4 months, and B put in \$350 for 2 months. They gained \$99, and A received for principal and gain \$136. How much stock did A put in?

17. A gentleman has a pleasure garden 80 rods long and 60 rods wide, surrounded by a walk of uniform breadth. The walk contains 576 square rods. Required the breadth of the walk.

18. A grocer filled a cask containing 40 gallons with wine. He then drew out a certain quantity and filled up the cask with water. After this he drew out the same quantity of liquid as before, and found that there remained in the cask only $22\frac{1}{2}$ gallons of pure wine. How many gallons of liquid were drawn out each time?

SECTION XXXIV.

EXTRACTION OF THE THIRD ROOTS OF NUMBERS.

Art. 98. Find three numbers which are to each other as 1, 2, and 3, and whose continued product is 384.

Let $x =$ the first number; then $2x =$ the 2d, and $3x =$ the 3d. Hence, $6x^3 = 384$.

In order to solve this equation, we divide by 6, and have $x^3 = 64$, or $xxx = 64$. We see now that x must be a number, which, multiplied twice by itself, will produce 64, and we find by trial that $x = 4$.

The numbers required, therefore, are 4, 8, and 12.

The equation arising from the preceding question, is called an equation of the third degree, because it involves the third power of the unknown quantity; and the process of finding the first power of a quantity, when the third is given, is called *extracting the third root.* The third root of any quantity, is that quantity which, being multiplied twice by itself, will produce the proposed quantity. Thus, 4 is the third root of 64; x is the third root of x^3.

To facilitate the extraction of the third roots of numbers, we shall give the third powers of those integral numbers, which consist of but one figure. They are as follows.

Roots.	1,	2,	3,	4,	5,	6,	7,	8,	9.
Third powers.	1,	8,	27,	64,	125,	216,	343,	512,	729.

The numbers in the 2d line are the 3d powers of those immediately over them, and the numbers in the first line are the third roots of those immediately beneath them.

The third power of $10 = 1000$; that of $100 = 1000000$, and that of $1000 = 1000000000$. Hence, the third power of an integral number between 10 and 100, that is, of a number consisting of two figures, must be between 1000 and 1000000; that is, it cannot contain less than four nor more than six figures; and

the third power of an integral number between 100 and 1000, or of a number consisting of three figures, must be between 1000000 and 1000000000; that is, it cannot contain less than seven nor more than nine figures. In like manner, it may be shown, that the third power of a number consisting of four figures, cannot contain less than ten nor more than twelve figures; and so on.

We can, therefore, immediately determine how many figures the root of any number will contain, by commencing at the right, and separating the number into portions, or periods, of three figures each. The left hand period may contain one, two, or three figures; and the root will contain as many figures as there are periods in the power. This separation may be denoted by accents, as in the extraction of the second root.

It appears also from the table given above, that, among integral numbers, consisting of one, two, or three figures, there are only nine which are exact third powers; consequently, the roots of the intermediate numbers cannot be obtained exactly, although they may be approximated, as we shall see hereafter, to any degree of exactness. Thus, the third root of 72 is between 4 and 5; the former being nearer the true root than the latter.

When a number consists of no more than three figures, provided it is a perfect third power, its root may be found immediately by inspection or trial; when there are more than three figures in the power, its root is, in some measure, obtained by trial, but a rule may be found which will greatly facilitate the operation.

Let us consider a number of more than three figures, as 50653, which is the third power of 37. Let $a =$ the tens and $b =$ the units of the root. Then, $a + b = 30 + 7 = 37$. The third power of $a + b$ is

$$a^3 + 3a^2b + 3ab^2 + b^3.$$

By putting 30 instead of a, and 7 instead of b, we have

$$a^3 = (30)^3 = 27000,$$
$$3a^2b = 3 \,.\, (30)^2 \,.\, 7 = 18900,$$
$$3ab^2 = 3 \,.\, 30 \,.\, (7)^2 = 4410,$$
$$b^3 = (7)^3 = 343.$$

Hence, $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 = 27000 + 18900 + 4410 + 343 = 50653$.

Therefore, the third power of a number consisting of tens and units, contains the third power of the tens, plus three times the second power of the tens into the units, plus three times the tens into the second power of the units, plus the third power of the units.

Now let it be proposed to find the third root of 50653, that root being supposed unknown.

Operation.

$$
\begin{array}{rl}
 & 50'653(\underline{37.}\quad \text{Root.} \\
 & \underline{27} \\
 & 23653(\underline{27.}\quad \text{Divisor.} \\
(37)^3 = & \underline{50653} \\
 & 0.
\end{array}
$$

Separating the number into periods, we see that the root must contain two figures, tens and units. The third power of the tens can contain no significant figure below thousands; it must therefore be found in the 50 (thousands). The greatest third power of tens contained in 50 (thousands), is 27 (thousands), the root of which is 3 (tens). Subtract 27 (thousands) from 50653, and there remains 23653.

This remainder contains $3\,a^2\,b + 3\,a\,b^2 + b^3$, or three times the second power of the tens into the units, three times the tens into the second power of the units, and the third power of the units.

If it contained exactly $3\,a^2\,b$, or three times the second power of the tens into the units, we should find b, or the units of the root, by dividing by $3\,a^2$, or three times the second power of the tens; but, as it contains something more than three times the second power of the tens into the units, if we divide by three times the second power of the tens, our quotient may be greater than the proper unit figure. Three times the second power of the tens or of 30, is 27 (hundreds), which is contained in 23653

eight times. If 8 be put with the 30 already found, it would make the root 38. But $(38)^3 = 54872$, which is greater than the given number. Therefore 8 is greater than the true unit figure. We next try 7, and find that $(37)^3 = 50653$. Hence, 37 is the third root of 50653.

As a second example, suppose it is required to find the third root of 43614208.

Operation.

$$
\begin{array}{rl}
 & 43'614'208(\underline{352}.\quad \text{Root.} \\
 & \underline{27\,.\,.\,.\,.\,.\,.} \\
\text{1st Dividend} = & 166\,.\,.\,.\,.\,.(\underline{27\,.\,.\,.\,.} = \text{1st Divisor.} \\
(35)^3 = & \underline{42875\,.\,.\,.} \\
\text{2d Dividend} = & \quad 7392\,.\,.(\underline{3675\,.\,.} = \text{2d Divisor.} \\
(352)^3 = & \underline{43614208} \\
 & \quad 0.
\end{array}
$$

Note. The points in the above operation are used to show the rank of the figures which precede them.

Separating the number into periods of three figures each, we see that the root must contain three figures, viz: hundreds, tens, and units. Let $a =$ the hundreds, $b =$ the tens, and $c =$ the units of this root. The third power of $a + b + c = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 + 3\,a^2\,c + 6\,a\,b\,c + 3\,b^2\,c + 3\,a\,c^2 + 3\,b\,c^2 + c^3$, or $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 + 3\,(a + b)^2\,c + 3\,(a + b)\,c^2 + c^3$; for, $3\,a^2\,c + 6\,a\,b\,c + 3\,b^2\,c = 3\,(a^2 + 2\,a\,b + b^2)\,c$, or $3\,(a + b)^2\,c$, and $3\,a\,c^2 + 3\,b\,c^2 = 3\,(a + b)\,c^2$.

We are first to seek the third power of the hundreds of the root, which must be found in the 43 (millions). The greatest third power of hundreds in 43 (millions) is 27 (millions), the root of which is 3 (hundreds), which we place at the right. Subtracting 27 (millions), or a^3, from the given number, we have for a remainder 16614208, or $3\,a^2\,b + 3\,a\,b^2 + b^3$, &c.

Our next object is to find b, or the tens of the root. The formula $3\,a^2\,b + 3\,a\,b^2 + b^3$ shows, that, if we divide by $3\,a^2$, or three times the second power of the hundreds, we shall obtain b,

the tens of the root, or a number a little too great. But since three times the second power of hundreds multiplied by tens, can produce no significant figure below hundreds of thousands, that is, below the sixth place from the right, it is sufficient to subtract a^3, or 27 (millions), from the first period, and to bring down, at the right of the remainder, the sixth figure, that is, the first figure of the next period.

Taking then 166, (16600000), and dividing it by three times the second power of 3 (hundreds), which is 27, (270000), the quotient would be 6 (tens); but that being found by trial to be too great, we take 5 (tens). Placing the 5 (tens) in the root at the right of the 3 (hundreds), we raise 35 (tens) to the third power, which gives 42875 (thousands), or $a^3 + 3a^2b + 3ab^2 + b^3$. This being subtracted from the given number, 43614208, leaves 739208, or $3(a+b)^2c + 3(a+b)c^2 + c^3$.

Now in order to obtain the units c of the root, we must evidently divide the remainder by $3(a+b)^2$, that is, by three times the second power of 35 (tens), which is 3675 (hundreds). But since the second power of tens multiplied by units, can have no significant figure below hundreds, that is, below the third figure from the right, it is sufficient to subtract the third power of 35 from the first two periods, and to the right of the remainder bring down the first figure of the next period to form a dividend. This dividend, 7392 (hundreds), divided by 3675 (hundreds), gives 2 for a quotient, which we place at the right of 35, and obtain 352 for the entire root. This root raised to the third power, gives 43614208, showing that 352 is the true root sought.

In the process just explained, it is necessary, after finding a new figure in the root, to raise the whole root, so far as it has been found, to the third power, and subtract the result from as many of the left hand periods, as there are figures already found in the root. But, by considering the local values of the figures, we may shorten the process of extracting the third root. To show the mode of doing this, we shall resume the last question.

Operation.

$$\begin{array}{lll}
43'614'208(352. & & \\
27\ldots\ldots & & = a^3. \\
\hline
166'14\ldots(27\ldots\ldots & & = 3\,a^2. \\
\qquad\qquad 45\ldots & & = 3\,a\,b. \\
\qquad\qquad\ \ 25\ldots & & = b^2. \\
\qquad\qquad 3175\ldots & & = 3\,a^2 + 3\,a\,b + b^2. \\
\qquad\qquad\quad 5\,. & & = b. \\
\hline
15875\ldots & & = 3\,a^2\,b + 3\,a\,b^2 + b^3. \\
\hline
\quad 7392'08(3675\ldots & & = 3\,(a+b)^2. \\
\qquad\qquad 210\,. & & = 3\,(a+b)\,c. \\
\qquad\qquad\qquad 4 & & = c^2. \\
\qquad\qquad 369604 & & = 3\,(a+b)^2 + 3\,(a+b)\,c + c^2. \\
\qquad\qquad\qquad 2 & & = c. \\
\quad 739208 & & = 3\,(a+b)^2\,c + 3\,(a+b)\,c^2 + c^3. \\
\hline
\qquad 0. & &
\end{array}$$

We proceed, until we find the second figure of the root, in the manner already explained; except that we annex to the remainder the whole of the second period, separating by an accent the two right hand figures. At this stage of the process, we have already subtracted the value of a^3, and our remainder with the second period annexed, contains $b\,(3\,a^2 + 3\,a\,b + b^2)$ and something more. We next wish to find the value of $b\,(3\,a^2 + 3\,a\,b + b^2)$, and subtract it from what remains of the first two periods, after the subtraction of a^3.

Now $3\,a^2 = 27$, (270000), is our divisor, and $3\,a\,b = 3 \times 3$ (hundreds) $\times$ 5 (tens) $=$ 45 (thousands), is three times the product of the last figure found and the preceding figure of the root; but as b is of the order of units next below a, the value of $3\,a\,b$ will contain a significant figure one degree lower than is found in the value of $3\,a^2$; therefore 45, $= 3\,a\,b$, is to be placed under 27, $= 3\,a^2$, one figure further to the right.

We now find $b^2 = 5$ (tens) $\times$ 5 (tens) $=$ 25 (hundreds), and as this contains a significant figure one degree lower than is

found in the value of $3ab$, it should be placed under this last, one place further to the right.

These three results being added as the figures now stand, will give 3175 (hundreds), $= 3a^2 + 3ab + b^2$, which multiplied by 5 (tens), $= b$, gives 15875 (thousands), $= 3a^2b + 3ab^2 + b^3$. Subtract this product from the dividend, including the two figures separated from the right, bring down the next period to the right of the remainder, and we have 739208, $= 3(a+b)^2c + 3(a+b)c^2 + c^3$.

Separating the two right hand figures, we take those remaining for a dividend, and find $3(a+b)^2$, $= 3$ times the square of 35 (tens), for a divisor; the resulting quotient is 2, $= c$, which we place in the root. We now multiply the preceding figures of the root by 2, $= c$, and take three times the product, which gives 210 (tens), $= 3(a+b)c$, and place the result under the divisor one figure further to the right, under which, one figure still further to the right, place 4, $= c^2$; adding these numbers as they stand, we have 369604, $= 3(a+b)^2 + 3(a+b)c + c^2$, which multiplied by 2, $= c$, gives 739208, $= 3(a+b)^2c + 3(a+b)c^2 + c^3$. This product subtracted from the last dividend, including the figures separated on the right, leaves no remainder; the work is therefore complete.

Art. **99**. From the preceding examples and explanations results the following

RULE FOR EXTRACTING THE THIRD ROOTS OF NUMBERS.

1. *Commencing at the right, separate the number into periods of three figures each; the left hand period may contain one, two or three figures.*

2. *Find the greatest third power in the left hand period, place its root at the right, and subtract the power from that period.*

3. *To the right of the remainder bring down the next period, separating by an accent the two right hand figures, and the result will form a dividend. For a divisor take three times the second power of the root already found. Divide the dividend by*

the divisor, rejecting the two figures of the former, before separated, and put the quotient as the second figure of the root.

4. *Take three times the product of the figure last found by the preceding part of the root, and place it under the divisor, one figure further to the right; under which, one figure further to the right, place the second power of the figure of the root last found. Add together the divisor and the numbers placed under it, as the figures stand, and multiply the sum by the figure of the root last found. Subtract this product from the dividend, including the two figures separated.*

5. *To the right of the remainder, bring down the next period, forming a new dividend, in the same manner as the first was formed. Take for a divisor three times the second power of the whole root so far as found; divide, rejecting the two right hand figures of the dividend, and place the quotient as the next figure of the root.*

6. *Find three times the product of the last figure by the whole of the preceding part of the root, and place it under the divisor one figure further to the right; under this, place, one figure further to the right, the second power of the last figure of the root found. Add the divisor and numbers placed under it, as the figures stand, multiply the sum by the last figure of the root found, and subtract the product from the dividend.*

7. *Repeat the operations stated in the 5th and 6th parts of the rule, until the given number is exhausted.*

Remark 1. If the divisor is not contained in the dividend, after the two right hand figures have been rejected, put a zero in the root, and bring down the next period; the divisor for finding the succeeding figure of the root, will then be the same as before, except with the addition of two zeros at the right.

Remark 2. If the number to be subtracted exceeds that from which it is to be taken, diminish the last figure found in the root, until a number is obtained which can be subtracted.

Art. **100**. We can always ascertain from the remainder, whether the figure last placed in the root, is so great as it should be. The third power of a is a^3, and that of $a+1$ is a^3+3a^2

$+3a+1$. Here the roots differ by 1, and the powers differ by $3a^2+3a+1$. Hence,

If the remainder after subtraction, contain three times the second power of the root already found, plus three times that root, plus 1, *or more, the last figure of the root is not sufficiently great by* 1 *at least.*

Thus, in the last example, if we had taken 4 instead of 5 for the second figure of the root, the remainder would have been 4310, which exceeds $3 \,.\, (34)^2 + 3 \,.\, 34 + 1$.

1. What is the third root of 127263527?

Operation.

```
127'263'527(503.  Root.
125
 22'63(75
 22635'27(7500
            450
              9
         754509
              3
 2263527
      0.
```

Find the third roots of the following numbers.

2. 300763.
3. 59319.
4. 753571.
5. 456533.
6. 37595375.
7. 48228544.
8. 751089429.
9. 27243729729.

SECTION XXXV.

THIRD ROOTS OF FRACTIONS — AND THE EXTRACTION OF THIRD ROOTS BY APPROXIMATION.

Art. **101**. A fraction is raised to the third power, by multiplying it twice by itself; but as fractions are multiplied together by taking the product of their numerators for a new numerator, and

that of their denominators for a new denominator, it follows that a fraction is raised to the third power by raising both numerator and denominator to the third power. For example, $\left(\frac{a}{b}\right)^3 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^3}{b^3}$.

Hence, the third root of a fraction is found by taking the third root of both numerator and denominator. Thus, the third root of $\frac{27}{125}$ is $\frac{3}{5}$; that of $\frac{a^3}{b^3}$ is $\frac{a}{b}$.

1. Find the third root of $\frac{343}{729}$.
2. Find the third root of $\frac{216}{2197}$.
3. Find the third root of $\frac{512}{1331}$.
4. Find the third root of $\frac{12167}{19683}$.
5. Find the third root of $83\frac{379}{512}$.

Art. **102.** When either the numerator or denominator is not an exact third power, the root of the fraction can be obtained only by approximation. For example, if the third root of $\frac{3}{7}$ be required, we may multiply both terms of the fraction by 49, the second power of the denominator 7, and the fraction becomes $\frac{147}{343}$. The denominator is now a perfect third power, the root of which is 7, and the nearest root of the numerator is 5. The approximate root, therefore, of $\frac{3}{7}$ is $\frac{5}{7}+$, which differs from the true root by less than $\frac{1}{7}$.

If a more accurate root be required, we may, after having multiplied both terms of the fraction by the second power of the denominator, multiply both terms of the result by any third power, and then find the nearest root. For example, $\frac{4}{5} = \frac{4 \cdot 5^2}{5 \cdot 5^2} = \frac{4 \cdot 5^2 \cdot 12^3}{5 \cdot 5^2 \cdot 12^3} = \frac{172800}{5^3 \cdot 12^3}$, the approximate third root of which is $\frac{56}{60}-$. This root is exact to within $\frac{1}{60}$, the product of $\frac{1}{5}$ by $\frac{1}{12}$.

What is the third root of $\frac{2}{3}$, accurate to within $\frac{1}{3} \cdot \frac{1}{15} = \frac{1}{45}$?

We have $\frac{2}{3} = \frac{2 \cdot 3^2}{3 \cdot 3^2} = \frac{2 \cdot 3^2 \cdot 15^3}{3 \cdot 3^2 \cdot 15^3} = \frac{60750}{3^3 \cdot 15^3}$, the root of which is $\frac{39}{45}+$.

Remark. To find the third root of a fraction to within a given limit, divide the denominator of the limit by that of the given fraction; then multiply both terms of the given fraction by the second power of its denominator and the third power of the quotient previously found; after which take the nearest root. Thus, in the last question, $\frac{1}{45}$ is the limit. The denominator 45 divided by 3, gives 15 for a quotient. We then multiply both terms of $\frac{2}{3}$ by 3^2 and 15^3.

1. Find the third root of $\frac{2}{7}$ to within $\frac{1}{63}$.
2. Find the third root of $\frac{2}{13}$ to within $\frac{1}{195}$.
3. Find the third root of $\frac{1}{11}$ to within $\frac{1}{187}$.

In a similar manner, we may approximate the third roots of whole numbers which are not perfect third powers, by converting them into fractions, whose denominators are third powers. Thus $2 = \frac{2 \cdot 12^3}{12^3} = \frac{3456}{12^3}$, the approximate root of which is $\frac{15}{12}+$, exact to within $\frac{1}{12}$.

Art. **103.** But the most convenient way to approximate the third root either of a whole number or a fraction, is to change it into a decimal, whose denominator is the third power of 10, 100, or 1000, &c., and take the root of the result. Thus, $3 = \frac{3 \cdot 10^3}{10^3} = \frac{3000}{1000}$, the root of which is $\frac{14}{10} + = 1{\cdot}4 +$. If a more accurate root is wanted, we may reduce 3 to a fraction whose denominator is the third power of 100; thus, $3 = \frac{3 \cdot 100^3}{100^3} = \frac{3000000}{1000000}$, the root of which is $\frac{144}{100} + = 1{\cdot}44 +$. Hence we see that three zeros are annexed for every additional decimal of the root. Indeed, this is evident; for the third power of ·1 is ·001, and the third power of ·01 is ·000001; thus, there are three times as many decimals in the power as there are in the root.

We may therefore omit the denominator, and merely annex three times as many zeros to the number as we wish to have decimals in the root. Nor is it necessary to add them all at

once, but only to annex three to the remainder, when a new figure of the root is required.

In like manner, to find the root of a vulgar fraction, convert it into a decimal, with thrice as many decimals as are required in the root.

If the number whose root is sought contain integers and decimals, and the number of decimals be not a multiple of three, make it so by annexing zeros to the right, which does not change the value, but only the denomination; or, point the number both ways from the decimal point, and then complete the right hand period, if necessary, by annexing zeros.

After these preparations, the third root of a number containing decimals, is found in the same way as that of an integral number, care being taken to point off one third as many decimals in the root as there are in the power.

Extract the third roots of the following numbers, finding three decimals in each root.

1. 2.
2. 7.
3. 115.
4. 1·5.
5. 25·7.
6. ·025.
7. 12·374.
8. 1256·4.
9. $\frac{7}{8}$.
10. $\frac{5}{11}$.
11. $7\frac{1}{2}$.
12. $12\frac{5}{7}$.
13. $3\frac{2}{3}$.
14. $\frac{2}{325}$.
15. $8\frac{41}{75}$.
16. $22\frac{1}{2}$.

SECTION XXXVI.

QUESTIONS PRODUCING PURE EQUATIONS OF THE THIRD DEGREE.

Art. **104.** A *pure equation of the third degree* contains the third power, but no other power, of the unknown quantity.

1. Three numbers are to each other as 2, 3, and 5; and their product is 240. What are these numbers?

2. A rectangular box contains 315 cubic feet. The breadth is $\frac{5}{9}$ and the depth is $\frac{7}{9}$ of the length. Required the three dimensions.

3. There are two numbers, such that the second power of the greater multiplied by the less is 500, and the second power of the less multiplied by the greater is 250. What are the numbers?

4. The depth of a cellar is to its length as 4 to 15, and the breadth is to the depth as 11 to 4; moreover, the cellar holds 5280 cubic feet. Required the three dimensions.

5. A pile of bricks is 8 feet high, 16 feet wide, and 32 feet long. What would be one of its sides, if it were in a cubical form?

6. A gentleman bought carpeting, sufficient to cover the floor of a square room, for $54. The carpet cost per square yard half as many shillings, as there were feet in one side of the room. Required the side of the room.

7. The less of two numbers is equal to one third of the sum of both; and the square of the greater multiplied by the less is 864. What are these numbers?

8. A bushel is $2150\frac{2}{5}$ cubic inches. Required one side of a cubical box, which shall contain 5 bushels.

9. The number of cubic feet in a pyramid is found, by multiplying together the number of square feet in the base, and one third of the altitude. If the base of a pyramid is a square, and the altitude is four times one side of the base, what is the altitude, and what is one side of the base of a pyramid, which contains 40000 cubic feet?

10. The solid contents of a cylinder are found, by taking the continued product of the length, the square of the radius of the base, and the number 3·14159. Required the radius of the base, and the length of a cylinder, if the length is to the radius as 7 to 2, and the cylinder contains 87·96452 cubic feet.

11. The solidity of a sphere is $\frac{4}{3}$ of 3·14159 multiplied by the cube of the radius. Required the radius of a sphere, which contains 28 cubic feet.

SECTION XXXVII.

POWERS OF MONOMIALS OR SIMPLE ALGEBRAIC QUANTITIES.

Art. **105.** Any power of a quantity may be found by multiplying it by itself a number of times denoted by the index of the power minus one. Thus, the second power of a or a^1 is $a \,.\, a = a^{1+1} = a^{1 \times 2} = a^2$, Art. **19**; the third power of a is $a \,.\, a \,.\, a = a^{1+1+1} = a^{1 \times 3} = a^3$; the fifth power of a is $a \,.\, a \,.\, a \,.\, a \,.\, a = a^{1+1+1+1+1} = a^{1 \times 5} = a^5$. The second power of $a\,b$ is $a\,b \,.\, a\,b = a^{1 \times 2}\, b^{1 \times 2} = a^2\, b^2$; the third power of $2\,b\,c$ is $2\,b\,c \,.\, 2\,b\,c \,.\, 2\,b\,c = 2^{1 \times 3}\, b^{1 \times 3}\, c^{1 \times 3} = 2^3\, b^3\, c^3 = 8\, b^3\, c^3$; and the fourth power of $4\, b^2\, c^3\, d^4$ is $= 4^4\, b^{2 \times 4}\, c^{3 \times 4}\, d^{4 \times 4} = 256\, b^8\, c^{12}\, d^{16}$. In these examples, adding the exponent of any quantity to itself, is the same as multiplying this exponent. Hence we have the following

RULE FOR RAISING A MONOMIAL TO ANY POWER.

Raise the numerical coëfficient to the required power, and multiply the exponent of each letter by the number which marks the degree of that power.

It is moreover manifest that any power of a product, is the product of that power of each of its factors. Thus, the fourth power of $4\, b^2\, c^3\, d^4$, which is $256\, b^8\, c^{12}\, d^{16}$, is the product of the fourth powers of 4, b^2, c^3, and d^4.

Remark. With regard to the signs, when the index of the power is even, the power will always have the sign $+$; but when the index is odd, the power will have the same sign as the root. This manifestly follows from the rules for multiplication.

1. Find the 2d power of $7\, a\, m^2$.
2. Find the 2d power of $8\, b^2\, c\, x^4$.
3. Find the 2d power of $15\, a^4\, m^3\, p^7$.
4. Find the 7th power of $2\, x^2\, y^3$.

5. Find the 13th power of $b^2 c^6 d^7$.
6. Find the 10th power of $2 b^3 c^2 d^9$.
7. Find the mth power of $p^4 q^5 x$. Ans. $p^{4m} q^{5m} x^m$.
8. Find the mth power of $p^n q^t$.
9. Find the mth power of $2 x^2 y^3$. Ans. $2^m x^{2m} y^{3m}$.

Remark. In the preceding example, since m is indefinite, the power of 2 must be represented merely.

10. Find the mth power of $3 p^x q^y$.
11. Find the 4th power of $-3 p^2 q^2$.
12. Find the 5th power of $-2 x^3 y^7$.
13. Find the 3d power of $-7 a^2 b^2 c d$.
14. Find the 6th power of $-2 a m^3 n^2 p^5 q^7 x y$.
15. Find the 2d power of $\frac{2a}{3b}$. Ans. $\frac{4a^2}{9b^2}$.
16. Find the 2d power of $\frac{3bc}{m^2}$.
17. Find the 3d power of $\frac{5m^2 n}{6x^3 y^2}$.
18. Find the 4th power of $\frac{7x}{21 p^2 x^3}$.

SECTION XXXVIII.

POWERS OF POLYNOMIALS.

Art. **106**. Any power of a polynomial may be *indicated* by enclosing it in a parenthesis, and placing the index of the power over it at the right. Thus, $(2b + c)^2$ represents the second power of $2b + c$. The same thing may be indicated by a vinculum and the exponent, thus, $\overline{2b + c}^2$.

Powers thus indicated may be raised to other powers in the same manner as simple quantities, that is, by multiplying the exponents. For example, the fourth power of $(a + b)^3$ is

$(a+b)^{3\times4}=(a+b)^{12}$. Moreover, when several compound quantities are represented as multiplied together, the whole is raised to any power by raising each factor to the power required. Thus, the second power of $(2c+d)(3m-n)^3$ is $(2c+d)^2\times(3m-n)^6$; the third power of $2a(b+c)^2(m+n)^5$ is $8a^3(b+c)^6(m+n)^{15}$. When some of the factors are monomials, they should be raised to the power required, in the manner already explained.

1. Indicate the 4th power of $6m-n+p$.
2. Indicate the 3d power of $(b+c+d)^6$.
3. Indicate the 7th power of $(4ab+4xy)^2$.
4. Indicate the 13th power of $(x-y)^2$.
5. Indicate the 2d power of $(a+b)(a-b)^3$.
6. Indicate the 5th power of $3(x-y)^3(p-q)^5$.
7. Indicate the 3d power of $2(a+b+c)^m$.
8. Indicate the 4th power of $am(c-d)^m(x+y)^n$.
9. Indicate the mth power of $(a+b+c)^3$.
10. Indicate the nth power of $(a+b)^2(c-d)^3$.
11. Indicate the mth power of $(x+2y)^n$.
12. Indicate the mth power of $(a+x)^p(n-x)^q$.
13. Indicate the 2d power of $\frac{a+b}{c+d}$. Ans. $\left(\frac{a+b}{c+d}\right)^2$.
14. Indicate the 3d power of $\frac{x^2-y^2}{b^2+n^2}$. Ans. $\left(\frac{x^2-y^2}{b^2+n^2}\right)^3$.
15. Indicate the 4th power of $\frac{b+c}{x+y}$.
16. Indicate the 3d power of $\frac{(m+n)^2}{(p+q)^3}$.
17. Indicate the 4th power of $\frac{2(a+b)(x-y)^2}{3(c+d)^8}$.
18. Indicate the 7th power of $\frac{a(m+n)^3(x-y)^5}{b(r+s)^2(a+b+c)^3}$.

Art. **107.** But if we would have the powers of polynomials in a developed form, they may be obtained by multiplication, in the manner of simple quantities. For example, $(m+n)^3=$

$(m+n)(m+n)(m+n)=m^3+3m^2n+3mn^2+n^3$; and $(2bc+d)^2=(2bc+d)(2bc+d)=4b^2c^2+4bcd+d^2$.

To develop the expression $a(b+c)^2$, we must first find the value of $(b+c)^2$, which is $b^2+2bc+c^2$, and then multiply by a, which gives $ab^2+2abc+ac^2$. If the multiplication had been performed before raising $b+c$ to the second power, the result would have been $a^2b^2+2a^2bc+a^2c^2$, which is erroneous.

1. Develop $(m-x)^2$.
2. Develop $c(a+b)^3$.
3. Develop $(a+b+c)^2(m+n)$.
4. Develop $a^3(x+y)^2$.
5. Develop $(2c+3d)^2$.
6. Develop $\frac{(a+b)^2}{(c+d)^3}$.

But the finding of the powers of polynomials by multiplication becomes, when the power is of a high degree, exceedingly tedious; and a more concise and expeditious method has been devised. The principle of this method is called the *Binomial Theorem*, and was discovered by Sir Isaac Newton. It is primarily applied to binomial quantities, but may be extended to polynomials.

Art. **108.** As a table of the powers of a binomial will sometimes be found convenient for the purpose of reference, we subjoin a few of the powers of $a+x$, obtained by multiplication.

$(a+x)^1=a+x$.
$(a+x)^2=a^2+2ax+x^2$.
$(a+x)^3=a^3+3a^2x+3ax^2+x^3$.
$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4$.
$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.

If the second term of the binomial is negative, the powers will be the same as when it is positive, except that the successive terms will be alternately positive and negative; that is, all the terms in which an odd power of the negative term enters, will be negative, all the others being positive. This follows from the

rules for multiplication; because, when the number of negative factors is even, the product is positive, but when the number of negative factors is odd, the product is negative. The first five powers of $a-x$, therefore, are as follows, viz:

$$(a-x)^1 = a - x.$$
$$(a-x)^2 = a^2 - 2\,a\,x + x^2.$$
$$(a-x)^3 = a^3 - 3\,a^2 x + 3\,a\,x^2 - x^3.$$
$$(a-x)^4 = a^4 - 4\,a^3 x + 6\,a^2 x^2 - 4\,a\,x^3 + x^4.$$
$$(a-x)^5 = a^5 - 5\,a^4 x + 10\,a^3 x^2 - 10\,a^2 x^3 + 5\,a\,x^4 - x^5.$$

SECTION XXXIX.

BINOMIAL THEOREM.

Art. **109**. The most concise demonstration of this theorem is that of *indeterminate coëfficients;* and, as subsidiary to the demonstration, we shall prove the following proposition, viz:

If, whatever be the value of x, (any indefinite quantity), two polynomials involving successive powers of x, as $\mathrm{A} + \mathrm{B}\,x + \mathrm{C}\,x^2 + \mathrm{D}\,x^3 + \mathrm{E}\,x^4$, *&c., and* $\mathrm{A}' + \mathrm{B}'\,x + \mathrm{C}'\,x^2 + \mathrm{D}'\,x^3 + \mathrm{E}'\,x^4$, *&c., are equal, we shall always have* $\mathrm{A} = \mathrm{A}'$, $\mathrm{B} = \mathrm{B}'$, $\mathrm{C} = \mathrm{C}'$, $\mathrm{D} = \mathrm{D}'$, $\mathrm{E} = \mathrm{E}'$, *&c.; that is, the terms which do not contain x are equal, as are also the coëfficients of the same powers of x.*

Since, in the equation $A + B\,x + Cx^2 + D\,x^3$, &c., $= A' + B'\,x + C'\,x^2 + D'\,x^3$, &c., the two members are equal, independently of x, they must be equal when $x = 0$; but, in this case, the terms all vanish except the first in each member, and the equation becomes $A = A'$. Subtracting these equal quantities, and dividing the remainders by x, we have

$$B + C\,x + D\,x^2, \text{\&c.} = B' + C'\,x + D'\,x^2, \text{\&c.}$$

Again suppose $x = 0$, and this last equation becomes $B = B'$. In like manner, it may be proved that $C = C'$, $D = D'$, &c.

Art. **110**. The binomial $x+a$ may be put under the form of $x\left(1+\frac{a}{x}\right)$, so that $(x+a)^m=x^m\left(1+\frac{a}{x}\right)^m$, Art. **105**. To avoid fractions, put $y=\frac{a}{x}$, and by substitution we have $(x+a)^m=x^m(1+y)^m$; hence, to obtain the value of the mth power of $x+a$, it will be sufficient to find that of $(1+y)^m$ developed, then restore the value of y, and multiply the whole by x^m.

From the manner in which a binomial is raised to any power by actual multiplication, it is manifest that $(1+y)^m$ developed, will be of the form of $A+By+Cy^2+Dy^3+Ey^4$, &c., in which the values of A and of the coëfficients B, C, D, &c., as well as the number of the terms, are wholly independent of the value of y, and are determined entirely by that of the exponent m. To make this more evident, we subjoin a few of the powers of $1+y$.

$$(1+y)^1=1+y.$$
$$(1+y)^2=1+2y+y^2.$$
$$(1+y)^3=1+3y+3y^2+y^3.$$
$$(1+y)^4=1+4y+6y^2+4y^3+y^4.$$
$$(1+y)^5=1+5y+10y^2+10y^3+5y^4+y^5.$$

We see, therefore, that in each power $A=1$, whereas B, the coëfficient of the second term, is different in different powers; the same is the case with C, &c., except with regard to the coëfficient of the last term, which is always 1. Moreover, in each power the number of terms exceeds by 1 the number which marks the degree of that power. Hence we infer that in the mth power there will be $m+1$ terms.

Art. **111**. Suppose, then,

$$[1]\quad (1+y)^m=A+By+Cy^2+Dy^3+Ey^4, \text{ \&c.},$$

in which the values of A, B, C, &c. are to be determined.

We have already inferred that A is always 1; this however may be demonstrated; for, since equation [1] is true for all values of y, it is true when $y=0$, which reduces the equation to

$(1)^m = A$. But, since every power of 1 is 1, we necessarily have $1 = A$.

We proceed now to investigate the other coëfficients B, C, D, &c. Since these coëfficients are entirely independent of the value of y, we have, in like manner,

[2] $(1+z)^m = A + Bz + Cz^2 + Dz^3 + Ez^4$, &c.

Hence, by subtraction and the union of terms which have a common coëfficient,

[3] $(1+y)^m - (1+z)^m = B(y-z) + C(y^2-z^2) + D(y^3-z^3) + E(y^4-z^4)$, &c.

Each of the factors $y-z$, y^2-z^2, &c., is divisible by $y-z$; actually dividing, therefore, the second member of equation [3] by $y-z$, and representing the division of the first member, we have

[4] $\dfrac{(1+y)^m-(1+z)^m}{y-z} = B + C(y+z) + D(y^2+yz+z^2) + E(y^3+y^2z+yz^2+z^3)$, &c.

Add $1-1$ to $y-z$, which does not change its value, and it becomes $y+1-z-1$, or $(1+y)-(1+z)$. The first member of equation [4] then becomes $\dfrac{(1+y)^m-(1+z)^m}{(1+y)-(1+z)}$. The division can now be performed, and gives, Art. **46**,

[5] $\dfrac{(1+y)^m-(1+z)^m}{(1+y)-(1+z)} = (1+y)^{m-1} + (1+y)^{m-2} \times (1+z) + (1+y)^{m-3}(1+z)^2 + (1+y)^{m-4}(1+z)^3 + \ldots \ldots + (1+z)^{m-1}$.

Suppose $y=z$, and substitute y for z in the second member of equation [5]; the terms then become alike, and, as the number of them is m, the sum of the whole is $m(1+y)^{m-1}$. Substitute this instead of the first member of equation [4], and, in the second member, put y instead of z and reduce, and we have

[6] $m(1+y)^{m-1} = B + 2Cy + 3Dy^2 + 4Ey^3$, &c.

Multiply both members by $1+y$, arrange the second member of the result according to the powers of y, and equation [6] becomes

[7] $m(1+y)^m = B + (B+2C)y + (2C+3D)y^2 + (3D+4E)y^3$, &c.

By substituting now for $(1+y)^m$, its value, given in equation [1], and putting 1 instead of A, equation [7] becomes
$m(1+By+Cy^2+Dy^3+Ey^4, \&c.) = B+(B+2C)y+(2C+3D)y^2+(3D+4E)y^3$, &c., or,

[8] $m+mBy+mCy^2+mDy^3+mEy^4, \&c. = B+(B+2C)y+(2C+3D)y^2+(3D+4E)y^3$, &c.

But, as was proved at the commencement of this section, the terms not involving y are equal, as are also the coëfficients of the same powers of y.

Hence, $B = m$;

$B+2C = mB$; hence, $C = \frac{B(m-1)}{2} = \frac{m(m-1)}{1.2}$;

$2C+3D = mC$; hence, $D = \frac{C(m-2)}{3} = \frac{m(m-1)(m-2)}{1.2.3}$;

$3D+4E = mD$; hence, $E = \frac{D(m-3)}{4} =$

$\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}$.

These results are sufficient to enable us to continue the formation of the coëfficients as far as we please. The next coëfficient would evidently be $\frac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5}$, and the succeeding one $\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{1.2.3\ 4.5.6}$.

Substitute these values of A, B, C, &c. in equation [1], and it becomes,

$$(1+y)^m = 1 + \frac{m}{1}y + \frac{m(m-1)}{1.2}y^2 + \frac{m(m-1)(m-2)}{1.2.3}y^3 + \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}y^4, \&c.$$

Restoring the value of y, viz: $y = \frac{a}{x}$, we have,

$$\left(1+\frac{a}{x}\right)^m = 1 + \frac{m}{1} \cdot \frac{a}{x} + \frac{m(m-1)}{1 \cdot 2} \cdot \frac{a^2}{x^2} +$$
$$\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cdot \frac{a^3}{x^3} + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{a^4}{x^4}, \text{ \&c.}$$

Multiplying both members by x^m,

$$x^m\left(1+\frac{a}{x}\right)^m \text{ or } (x+a)^m = x^m + \frac{m}{1} \cdot \frac{x^m a}{x} + \frac{m(m-1)}{1 \cdot 2} \cdot \frac{x^m a^2}{x^2}$$
$$+ \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cdot \frac{x^m a^3}{x^3} + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} \times$$
$$\frac{x^m a^4}{x^4}, \text{ \&c.}$$

Reducing the fractions to lower terms,

$$(x+a)^m = x^m + m x^{m-1} a + \frac{m(m-1)}{1 \cdot 2} x^{m-2} a^2 +$$
$$\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^{m-3} a^3 + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} \times$$
$$x^{m-4} a^4, \text{ \&c.}$$

Art. **112**. Such is the formula for any power of a binomial; from which we readily deduce the law both of the letters and the coëfficients.

First, with regard to the letters, we see that, in the first term, x, which is the first term of the binomial, is raised to the power to which the binomial was to be raised, and that the powers of x in the successive terms go on decreasing by unity.

Secondly, a, the second term of the binomial, is found in the second term of the power with 1 for its exponent, and, in the successive terms, the powers of a go on increasing by unity.

Moreover, the sum of the exponents of x and a in the same term, is always equal to m, the exponent of the power to which the binomial is raised.

With regard to the coëfficients; we perceive, that the coëfficient of the first term is 1; that of the second term is equal to m, the exponent of the power to which the binomial is raised.

To obtain the coëfficient of the third term, we multiply that of the second, which is m, by $\frac{m-1}{2}$; that is, we multiply by $m-1$ and divide by 2.

To obtain the coëfficient of the fourth term, we multiply that of the third by $\frac{m-2}{3}$; that is, multiply by $m-2$ and divide by 3, and so on.

Art. **113**. Hence, having one term of any power of a binomial, the succeeding term may be found by the following

RULE.

Multiply the given term by the exponent of x in that term, that is, by the exponent of the first or leading quantity of the binomial, and divide the product by the number which marks the place of the given term from the first inclusive; diminish the exponent of x by 1, *and increase that of a by* 1.

The coëfficient of the first term always being 1, and that of the second being the same as the index of the power required, we can, by the preceding rule, write any power of a binomial.

Let it be required, for example, to find the 9th power of $x+a$.

The first term is x^9; the second is $9\,x^8\,a$; the third is found by multiplying 9, the coëfficient of the second, by 8, the exponent of x in the same, dividing the product by 2, which marks the place of the second term, diminishing the exponent of x and increasing that of a each by unity. The third term then is $\frac{9\,.\,8}{2}\,x^7\,a^2 = 9\,.\,4\;x^7\,a^2 = 36\,x^7\,a^2$. The fourth term is $\frac{36\,.\,7}{3}\,x^6\,a^3 = 12\,.\,7\,x^6\,a^3 = 84\,x^6\,a^3$. Finding. in a similar manner, the succeeding terms, we have

$$(x+a)^9 = x^9 + 9\,x^8\,a + 36\,x^7\,a^2 + 84\,x^6\,a^3 + 126\,x^5\,a^4 + 126\,x^4\,a^5 + 84\,x^3\,a^6 + 36\,x^2\,a^7 + 9\,x\,a^8 + a^9.$$

Since any quantity with zero for an exponent is 1, we may suppose a^0 to enter into the first term, and x^0 into the last. If

we should attempt to find another term succeeding a^9 or $x^0 a^9$, we should obtain for its coëfficient $\frac{1 \,.\, 0}{10} = \frac{0}{10} = 0$. No additional terms therefore can be obtained.

Applying the rule and remembering that odd powers of negative quantities are negative, we have also

$(a - b)^{10} = a^{10} - 10\, a^9 b + 45\, a^8 b^2 - 120\, a^7 b^3 + 210\, a^6 b^4 - 252\, a^5 b^5 + 210\, a^4 b^6 - 120\, a^3 b^7 + 45\, a^2 b^8 - 10\, a b^9 + b^{10}$.

From the preceding examples, as well as from the table of powers given in the Art. **108**, we infer,

1. That the number of terms in each power of a binomial exceeds by 1 the index of that power. Thus, in the fifth power, there are six terms; in the ninth power, there are ten terms.

2. When the number of terms is odd, there is one coëfficient, in the middle of the series, greater than any of the others; but, when the number of terms is even, there are two coëfficients in the middle, of equal value and greater than any of the others. Moreover, those which precede and those which succeed the greatest or greatest two, are the same, only arranged in an inverse order.

Therefore, when half, or one more than half of the coëfficients have been found, the others may be written down without the trouble of calculation.

1. Find the seventh power of $a + b$.
2. Find the tenth power of $x + y$.
3. Find the fifth power of $m - n$.
4. Find the eleventh power of $b + c$.
5. Find the thirteenth power of $x - y$.
6. Find the sixth power of $3\, a + b$.

In this last example, the numerical coëfficient of a must be raised to the requisite powers by multiplication.

First write the power, merely indicating the operations with regard to $3\, a$, and we have

$(3\, a)^6 + 6\, (3\, a)^5 b + 15\, (3\, a)^4 b^2 + 20\, (3\, a)^3 b^3 + 15\, (3\, a)^2 b^4 + 6\, (3\, a)\, b^5 + b^6$.

Raising $3a$ to the several powers indicated, and substituting the results,

$729a^6 + 6 . 243a^5b + 15 . 81a^4b^2 + 20 . 27a^3b^3 + 15 . 9a^2b^4 + 6 . 3ab^5 + b^6$.

Performing the multiplication, we have for the final result,

$729a^6 + 1458a^5b + 1215a^4b^2 + 540a^3b^3 + 135a^2b^4 + 18ab^5 + b^6$.

7. Find the fifth power of $x + 2y$.
8. Find the third power of $6a + 5x$.
9. Find the fourth power of $a + b - 2c$.

When a quantity containing several terms, as $a + b - 2c$, is to be raised to a power, it is convenient to substitute other letters, so as to render the quantity a binomial, raise this binomial to the required power, and then restore the value of the letters substituted.

Thus, in the present example, let $b - 2c = m$; then $a + b - 2c = a + m$. Now $(a + m)^4 = a^4 + 4a^3m + 6a^2m^2 + 4am^3 + m^4$. But,

$m = b - 2c$;

$m^2 = (b - 2c)^2 = b^2 - 4bc + 4c^2$;

$m^3 = (b - 2c)^3 = b^3 - 3b^2(2c) + 3b(2c)^2 - (2c)^3$, or,

$m^3 = b^3 - 6b^2c + 12bc^2 - 8c^3$;

$m^4 = (b - 2c)^4 = b^4 - 4b^3(2c) + 6b^2(2c)^2 - 4b(2c)^3 + (2c)^4$, or,

$m^4 = b^4 - 8b^3c + 24b^2c^2 - 32bc^3 + 16c^4$.

Putting these values instead of m, m^2, &c., and performing the multiplication by $4a^3$, $6a^2$, &c., we have

$(a + b - 2c)^4 = a^4 + 4a^3b - 8a^3c + 6a^2b^2 - 24a^2bc + 24a^2c^2 + 4ab^3 - 24ab^2c + 48abc^2 - 32ac^3 + b^4 - 8b^3c + 24b^2c^2 - 32bc^3 + 16c^4$.

10. Find the fifth power of $2a + 3x$.
11. Find the third power of $4x - 3y + a$.
12. Find the third power of $a + b + c + d$.

Let $a + b = m$, and $c + d = n$; then $a + b + c + d = m + n$.

13. Find the sixth power of $a + 2b - c$.

14. Find the fifth power of $a + b - 2c - 3d$.

In this example, let $a + b = m$, and $2c + 3d = n$; then $a + b - 2c - 3d = m - n$.

SECTION XL.

ROOTS OF NUMBERS TO ANY DEGREE.

Art. **114.** The different powers of a binomial suggest the means of extracting roots to any degree, both of numerical and literal quantities.

Let it be required, for instance, to find the fifth root of 9765625.

Operation.

$$\begin{array}{ll} 97'65625 & (\underline{25} = a + b \\ \underline{32\ldots\ldots} & \quad\ = a^5. \\ 656 & (\overline{80} = 5a^4. \\ 9765625 = (25)^5 = (a+b)^5. & \end{array}$$

As the fifth power of 10 is 100000, consisting of six figures, and that of 100 is 10000000000, consisting of eleven figures, the fifth root of 9765625 must be between 10 and 100; that is, it must consist of two figures, tens and units. Let a represent the tens and b the units of the root. The formula for the fifth power of a binomial is $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2$, &c. This shows us that we are first to seek the fifth power of the tens, which must be found in the 97, (9700000); or, what is the same thing, we are simply to seek the greatest fifth power in 97. Now $2^5 = 32$, and $3^5 = 243$. The greatest fifth power, therefore, in 97 is 32, the root of which is 2. Place 2 as the first figure or tens of the root, subtract 32 from 97, to the remainder annex the rest of the figures, and we have 6565625.

This remainder contains $5\,a^4\,b + 10\,a^3\,b^2$, &c., or five times the fourth power of the tens into the units, and something more. If, therefore, we divide by five times the fourth power of the tens, the quotient will be the units, or a number a little too great. But, as the fourth power of tens into units, can contain no significant figure below the fifth from the right, it is sufficient, after having subtracted 32 from 97, to bring down 6, the next figure, to the right of the remainder, and to take 656 for our dividend. Five times $2^4 = 80$, which is contained in 656 eight times. But if we put 8 in the root, at the right of the 2, and raise 28 to the fifth power, the result will be greater than 9765625. The same would be the case with 27 and 26. But if we take 5 as the unit figure, we find $(25)^5 = 9765625$. Therefore 25 is the true root.

If there were more than ten figures in the given number, there would be more than two in its root. We should, in that case, let a at first represent the highest order of units and b all the rest, until we found the second figure of the root; after which a might represent the two figures found and b the rest, and so on.

Moreover, it is easy to see, that the number is to be separated into periods of five figures each, except that the left hand period may contain less than five; that the root will contain as many figures as there are periods; that the fifth power of the first figure is to be subtracted from the first period, the fifth power of the first two, from the first two periods, that of the first three, from the first three periods, &c.; and, that, in each case, we are to take the remainder with the first figure of the next period for a dividend, and five times the fourth power of the figures already found for a divisor.

Similar explanations might be given for the extraction of fourth, sixth, seventh and other roots. The mode of procedure, in each case, may readily be deduced from the formulæ.

Art. **115.** We may, therefore, take the general formula for the binomial theorem, and deduce from it a rule for extracting roots of any degree whatever. This formula is

$a^m + m\,a^{m-1}b + \frac{m(m-1)}{1\,.\,2}a^{m-2}b^2$, &c., in which m denotes the degree of the root to be found. The first two terms alone, in connection with inferences which are easily drawn from what precedes, determine the rule, which is as follows.

RULE FOR EXTRACTING THE *m*TH ROOT OF A NUMBER

1. *Beginning at the right, separate the number into periods of* m *figures each; the left hand period may contain from one to* m *figures.*

2. *Find the greatest* m*th power in the left hand period, and put its root at the right of the given number, as the first figure of the required root. Subtract the* m*th power of this figure from the first period, and to the right of the remainder bring down the first figure of the next period to form a dividend.*

3. *For a divisor, take* m *times the* (m — 1)*th power of the root already found. Divide and place the quotient as the second figure of the root.*

4. *Raise these two figures to the* m*th power, and if the result does not exceed the first two periods of the number, subtract it from these two periods, and to the remainder annex the first figure of the succeeding period to form a new dividend. But, if the* m*th power of the first two figures exceeds the corresponding periods, diminish the second figure of the root, until an* m*th power is obtained which can be subtracted.*

5. *For a new divisor, take* m *times the* (m — 1)*th power of the whole root already found. The division will enable us to find the third figure of the root. Then raise the three figures to the* m*th power, and subtract the result from the first three periods; and thus proceed until all the periods have been used.*

Remark 1st. It is manifest that the second and third roots may be extracted according to the above rule, as well as according to the rules previously given. The particular rules are preferable, only because they render the operations shorter than the general rule would.

Remark 2d. When the number expressing the degree of the root, can be separated into factors, this may be done, and we

may find successively roots, the degrees of which are denoted by these factors. Thus, instead of finding the fourth root immediately, we may first find the second root, and then the second root of that result. For example, the second root of a^4 is a^2, and the second root of a^2 is a. In like manner, to obtain the sixth root, first find the second root, and then the third root of that result. To obtain the eighth root, extract the second root three times, and to get the ninth root, extract the third root twice.

1. Find the fourth root of 625.
2. Find the fourth root of 20736.
3. Find the fourth root of 28398241.
4. Find the fifth root of 2073071593.
5. Find the fifth root of 41·8227202051.

Remark. Point off both ways from the decimal point.

6. Find the sixth root of 4826809.

SECTION XLI.

ROOTS OF MONOMIALS OR SIMPLE ALGEBRAIC QUANTITIES.

Art. **116**. From the method given in Art. **105**, for obtaining powers of monomials, results the following

RULE FOR FINDING THE ROOT OF ANY MONOMIAL.

Extract the root of the numerical coëfficient, and divide the exponent of each literal factor by the number which marks the degree of the root.

The reason for this rule is manifest, since extracting a root is the reverse of finding a power. Thus, the second power of $5\,a\,b$ is $25\,a^2\,b^2$; consequently, the second root of $25\,a^2\,b^2$ is $5\,a^{\frac{2}{2}}\,b^{\frac{2}{2}} = 5\,a^1\,b^1$ or $5\,a\,b$. In like manner, the third root of $125\,a^6\,b^9\,c^3$ is $5\,a^2\,b^3\,c$.

Art. **117.** With regard to the signs which affect the roots of monomials, observe, that

Every root of an even degree may have either the sign $+$ *or* $-$. This is manifest from the formation of powers. Thus, the fourth power of $+a$ is $+a^4$, and the fourth power of $-a$ is also $+a^4$. Therefore the fourth root of $+a^4$ is either $+a$ or $-a$. Hence, to any root of an even degree, we commonly prefix $\pm$.

But roots of an odd degree have the same sign as the power. The third power of $+a$ is $+a^3$; whereas, the third power of $-a$ is $-a^3$; $+a$ is therefore the third root of $+a^3$, and $-a$, that of $-a^3$.

It has already been stated in Art. **88**, that the second root of a negative quantity is imaginary. The same is the case with any *even* root of a negative quantity. Thus, $(-16)^{\frac{1}{4}}$, $(-a)^{\frac{1}{6}}$, $(-a)^{\frac{1}{8}}$, or the equivalent expressions, $\sqrt[4]{-16}$, $\sqrt[6]{-a}$, $\sqrt[8]{-a}$, are *imaginary quantities;* for no quantity raised to a power of an even degree, can produce a negative quantity.

1. Find the second root of $a^4 m^6 x^2$.
2. Find the second root of $64 x^2 y^2$.
3. Find the third root of $343 a^9 p^3 q^6$.
4. Find the third root of $-729 a^6 b^3 c^{12}$.
5. Find the fourth root of $16 a^8 b^{16} c^{12}$.
6. Find the second root of $\dfrac{4 a^2}{25 x^4}$.
7. Find the third root of $\dfrac{27 x^9 y^3}{64 m^{12}}$.
8. Find the fifth root of $-\dfrac{32 a^{10} b^{15}}{3125 x^5 y^{10}}$.
9. Find the seventh root of $\dfrac{2187 a^{14} b^{21} c^7}{16384 x^{28} y^{35}}$.

The preceding examples, as well as what was said in Art. **105**, relative to the powers of products, show, *that any root of a product will be the product of the roots, to the same degree, of each of the factors of this product.* Thus, the second root of

$a^2 b^2 c^4$ is $a b c^2$, which is the product of the second roots of a^2, b^2 and c^4, the factors of $a^2 b^2 c^4$.

In like manner, if any numerical quantity is divided into factors which are exact powers of the required degree, (and this may always be done, when the number itself is an exact power of that degree,) we may extract separately the roots of these factors, and then multiply these roots together. For example, $1764 = 36 . 49$, the second root of which is $6 . 7 = 42$.

Art. **118.** From the preceding mode of finding the roots of literal quantities, it follows, that, if the exponent of any factor is not divisible by the number which expresses the degree of the root, the division can be expressed only, and gives rise to fractional exponents. Thus, the second root of a is $a^{\frac{1}{2}}$; the third root of a is $a^{\frac{1}{3}}$; the fourth root of a^3 is $a^{\frac{3}{4}}$.

The expression $a^{\frac{3}{4}}$ indicates either the fourth root of a^3 or the third power of $a^{\frac{1}{4}}$; for the third power of $a^{\frac{1}{4}}$ is $a^{\frac{1}{4} \times 3} = a^{\frac{3}{4}}$. In like manner, $a^{\frac{4}{5}}$ denotes either the fourth power of $a^{\frac{1}{5}}$ or the fifth root of a^4.

The radical sign may be used to indicate a root of any degree, if we place over it a figure denoting the degree of that root. Thus, $\sqrt{}$ denotes the second root; $\sqrt[3]{}$, the third root; $\sqrt[4]{}$, the fourth root, and so on. Hence,

$$\sqrt{a} = a^{\frac{1}{2}},$$
$$\sqrt[3]{a^2} = a^{\frac{2}{3}},$$
$$\sqrt[4]{a^3} = a^{\frac{3}{4}},$$
$$\sqrt[5]{a^6} = a^{\frac{6}{5}}.$$

Observe, that $\sqrt{a}$ is the same as $\sqrt[2]{a}$, the 2 over the sign being generally understood. We see, therefore, that in the preceding equivalent expressions, the number over the radical sign is the same as the denominator, and the exponent under the sign is the same as the numerator, of the fractional exponent.

Art. **119.** By means of exponents either entire or fractional, any quantity may be expressed in a great variety of forms. Thus, $a^3 = a^2 \,.\, a = a \,.\, a \,.\, a = a^2 \,.\, a^{\frac{1}{2}} \,.\, a^{\frac{1}{2}} = a \,.\, a^{\frac{1}{2}} \,.\, a^{\frac{1}{2}} \,.\, a^{\frac{1}{2}} \,.\, a^{\frac{1}{2}} = a^{\frac{3}{5}} \,.\, a^{\frac{2}{5}} \,.\, a^{\frac{1}{3}} \,.\, a^{\frac{2}{3}} \,.\, a^{\frac{1}{2}} \,.\, a^{\frac{1}{2}}$, &c. Also, $\sqrt[5]{a^6 b^2} = a^{\frac{6}{5}} \; b^{\frac{2}{5}} = a^{1\frac{1}{5}} \; b^{\frac{2}{5}} = a \,.\, a^{\frac{1}{5}} \,.\, b^{\frac{1}{5}} \,.\, b^{\frac{1}{5}} = a^{\frac{3}{8}} \,.\, a^{\frac{5}{8}} \,.\, a^{\frac{1}{5}} \,.\, b^{\frac{1}{5}} \,.\, b^{\frac{1}{5}}$, &c.

Hence, any quantity may be separated into an indefinite number of factors; the only restriction is, that the sum of the exponents of those factors, which are alike except with regard to their exponents, shall be the same as in the given quantity. In the first example given above, the sum of the exponents must be uniformly 3; in the second, the sum of the exponents of a must be $\frac{6}{5}$, and that of the exponents of b must be $\frac{2}{5}$.

1. Separate a^2 into three factors.
2. Separate a^4 into seven factors.
3. Separate a^3 into six factors.
4. Separate a into three factors.
5. Separate $a^{\frac{2}{3}} b$ into four factors.
6. Separate $3 a^2$ into six factors.
7. Separate 35 into three factors.
8. Separate 10 into seven factors.

SECTION XLII.

ROOTS OF POLYNOMIALS.

Art. **120.** Let it be required to find the second root of $4 m^2 + 12 m n + 9 n^2$.

Operation.

$$
\begin{array}{l}
4 m^2 + 12 m n + 9 n^2 \,(\underline{2 m + 3 n.} \quad \text{Root.} \\
\underline{4 m^2 \qquad\qquad\qquad\quad} \\
\qquad\quad 12 m n + 9 n^2 \,(\underline{4 m + 3 n.} \\
\qquad\quad \underline{12 m n + 9 n^2} \\
\qquad\qquad\quad 0.
\end{array}
$$

By recurring to the second power of $a+b$, which is $a^2+2ab+b^2$, we see that $4m^2$ corresponds to a^2. We therefore take the second root of $4m^2$, which is $2m$, and place it at the right, as the first term of the root sought, and subtract its second power from the given quantity. The remainder, $12mn+9n^2$, answers to $2ab+b^2$, or $(2a+b)b$. Dividing the first term of this remainder by $4m$, corresponding to $2a$, we have $3n$ for the second term of the root, which we annex to the $2m$ in the root, and also to the divisor. The divisor thus increased, becomes $4m+3n, = 2a+b$. We then multiply $4m+3n$ by $3n, = b$, and we have $12mn+9n^2$, which subtracted from the dividend, leaves no remainder. Hence, the second root of $4m^2+12mn+9n^2$ is $2m+3n$, or $-2m-3n$; or rather, $\pm 2m \pm 3n$.

The double sign may be omitted, until the operation is completed, and then all the signs of the root may be changed, if both roots are required.

When there are more than three terms in the power, the second root will contain more than two terms. But the mode of proceeding will be almost the same as that for finding the second roots of numbers. We form a second dividend, in the same manner as the first was formed, and for a divisor double the whole root found. The division will give the third term of the root. The process is manifest from the formula, $(a+b+c+d, \&c.)^2 = a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2$, &c., in which a, b, c, &c. represent the terms of the root.

The following example will serve to illustrate the process.

Required the second root of $a^4+6a^3x+11a^2x^2+6ax^3+x^4$.

Operation.

$$
\begin{array}{l}
a^4+6a^3x+11a^2x^2+6ax^3+x^4\ \underline{(a^2+3ax+x^2.}\\
\underline{a^4\qquad\qquad\qquad\qquad\qquad\qquad}\\
\qquad 6a^3x+11a^2x^2+6ax^3+x^4\ \underline{(2a^2+3ax.}\\
\qquad \underline{6a^3x+\ \ 9a^2x^2\qquad\qquad\qquad}\\
\qquad\qquad\qquad 2a^2x^2+6ax^3+x^4\ \underline{(2a^2+6ax+x^2.}\\
\qquad\qquad\qquad \underline{2a^2x^2+6ax^3+x^4}\\
\qquad\qquad\qquad\qquad\qquad 0.
\end{array}
$$

The root required is $a^2+3ax+x^2$.

From the preceding analysis we derive the following

RULE FOR EXTRACTING THE SECOND ROOT OF A POLYNOMIAL.

1. *Arrange the quantity according to the powers of some letter.*

2. *Find the root of the first term, and place it as the first term of the root sought, subtract the second power of this term from the given polynomial, and call the remainder the first dividend.*

3. *Double the term of the root found, for a divisor, by which divide the first term of the dividend, and place the quotient, with its proper sign, as the second term of the root, also at the right of the divisor. Multiply the divisor, with the term annexed, by the second term of the root, and subtract the product from the dividend.*

4. *The remainder will form a new dividend, which is to be divided by twice the whole root found, and the quotient is to be placed as the next term of the root, also at the right of the divisor. Multiply the divisor, with the term last annexed, by the last term of the root, and subtract the product from the last dividend.*

5. *The remainder will form a new dividend, with which proceed as before; and thus continue, until all the terms of the root are found.*

Remark 1. Each of the remainders must be arranged in the same order as the given polynomial was first arranged.

If the given quantity contains no fractions, and a dividend occurs, the first term of which does not contain all the letters of the first term of the divisor, or which contains any one of them with a less exponent than it has in that term of the divisor, we may be assured that the given quantity is not an exact second power, and, therefore, does not admit of an exact root.

Remark 2. In dividing we merely divide the first term of the dividend by the first term of the divisor; and, since double the first, the first two, the first three, &c. terms of the root, will have the first terms alike, it is manifest that the successive divisors will have their first terms the same.

Find the second roots of the following quantities.

1. $34x^2+9x^4+20x+12x^3+25.$
2. $a^4+54a^2b^2+12a^3b+108ab^3+81b^4.$
3. $10x^4-10x^3-12x^5+5x^2+9x^6-2x+1.$
4. $9a^4-20ab^3-12a^3b+34a^2b^2+25b^4.$
5. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4.$
6. $x^6+4x^5+2x^4+9x^2-4x+4.$
7. $4x^4+6x^3+\frac{89}{4}x^2+15x+25.$

Art. **121**. The rule for extracting the third roots of numbers, might, with slight modifications, be applied to the extraction of the third roots of algebraic polynomials. But it is generally the most convenient to use the rule, derived from the binomial theorem, for the extraction of roots to any degree. This rule, applied to literal quantities, will, as is evident from the formula for the mth power of $x+a$, be as follows.

RULE FOR EXTRACTING ANY ROOT OF A POLYNOMIAL.

1. *Arrange the quantity according to the powers of some letter.*

2. *Find the* m*th root of the first term, place it as the first term of the root sought, and subtract the* m*th power of it from the polynomial.*

3. *The remainder will form a dividend, which is to be divided by* m *times the* (m — 1)*th power of the term of the root found, and the quotient is to be placed as the second term of the root.*

4. *Raise the whole root to the* m*th power, and subtract the result from the given polynomial.*

5. *The remainder will form a new dividend, which is to be divided by* m *times the* (m — 1)*th power of the whole root already found, and the quotient placed as the third term of the root.*

6. *Raise the whole root to the* m*th power, subtract the result from the given polynomial, and with the remainder proceed as before; and thus continue until all the terms of the root are found.*

Remark. It is manifest that the first term of each successive divisor will be the same; and, since we always divide the first term of the dividend by the first of the divisor, it is sufficient to find the first term of the first divisor and use that throughout;

and, in subtracting, only one term of the remainder needs to be brought down, viz: that which contains the highest power of the letter according to which the given quantity was arranged.

As an example, let it be required to extract the third root of $8x^3 + 60x^2y + 150xy^2 + 125y^3$.

Operation.

$$\begin{array}{l} 8x^3 + 60x^2y + 150xy^2 + 125y^3 \;(\underline{2x + 5y}. \quad \text{Root.} \\ \underline{8x^3 \qquad\qquad\qquad} \\ \qquad\quad 60x^2y \;(\underline{12x^2}. \quad \text{Divisor.} \end{array}$$

The index m of the general formula, when applied to this question, is 3; and, after having arranged the quantity according to the powers of x, we find the third root of $8x^3$, which is $2x$; subtracting the third power of $2x$, we have, for the first term of the remainder, $60x^2y$, which we divide by $12x^2$, $=$ three times the second power of $2x$. The quotient is $5y$, which we put as the second term of the root, and raise $2x + 5y$ to the third power.* The result is the same as the given quantity, and, when subtracted, leaves no remainder. Therefore, $2x + 5y$ is the root sought.

As a second example, we shall trace the operations for extracting a root consisting of three terms.

Let it be required to extract the fifth root of $x^{10} - 10x^9a + 45x^8a^2 - 120x^7a^3 + 210x^6a^4 - 252x^5a^5 + 210x^4a^6 - 120x^3a^7 + 45x^2a^8 - 10xa^9 + a^{10}$.

The quantity being arranged according to the powers of x, we find the 5th root of x^{10}, which is x^2, and subtract the 5th power of this root from the given quantity. The first term of the remainder is $-10x^9a$. This term we divide by five times the 4th power of x^2, which is $5x^8$. The quotient, $-2ax$, we place as the second term of the root, and raise $x^2 - 2ax$ to the 5th power. The 5th power of $x^2 - 2ax$ is $x^{10} - 10x^9a + 40x^8a^2 - 80x^7a^3 + 80x^6a^4 - 32x^5a^5$, which subtracted from the given

* Let the learner use the binomial theorem for finding the powers of any quantity consisting of more than one term.

quantity, gives a remainder the first term of which is $5x^8a^2$. This term being divided by $5x^8$, the first term of five times the 4th power of x^2-2ax, gives for a quotient a^2, which we place as the third term of the root. We then raise $x^2-2ax+a^2$ to the 5th power, and it produces the whole of the given quantity. Hence, $x^2-2ax+a^2$ is the root sought.

1. Find the 3d root of $27a^3+81ax^2+81a^2x+27x^3$.
2. Find the 4th root of $16x^{12}+1000x^3a^6+600x^6a^4+160x^9a^2+625a^8$.
3. Find the 4th root of $625c^8-1000c^6yz+600c^4y^2z^2-160c^2y^3z^3+16y^4z^4$.
4. Find the 5th root of $32a^{10}-80a^8b^3+80a^6b^6-40a^4b^9+10a^2b^{12}-b^{15}$.
5. Find the 6th root of $729x^6+2916x^5y+4860x^4y^2+4320x^3y^3+2160x^2y^4+576xy^5+64y^6$.

SECTION XLIII.

SIMPLIFICATION OF IRRATIONAL OR RADICAL QUANTITIES.

Art. **122**. When a quantity is not an exact power of the degree required, its root cannot be found exactly. In such a case, the root is commonly expressed either by a radical sign or by a fractional exponent. Expressions indicating roots which cannot be accurately obtained, are called, as has already been stated, *irrational* or *incommensurable quantities*. They are also sometimes called *surds* or simply *radical quantities*. Thus, $\sqrt{2}$ or $2^{\frac{1}{2}}$ and $\sqrt[3]{4}$ or $4^{\frac{1}{3}}$ are irrational quantities.

In like manner, we are obliged to express the second root of a by a sign, thus $\sqrt{a}$ or $a^{\frac{1}{2}}$; although, perhaps, when a has been replaced by its numerical value, the root may be exactly found. Algebraically considered, however, such expressions are in the condition of irrational quantities.

Expressions of this kind may, in many cases, be simplified. The root of a product, as was shown in Art. **117**, is formed by multiplying together the roots of all the factors of that product. Hence, we may take the roots of such factors as are exact powers, and indicate the roots of the other factors, leaving these roots to be approximated afterwards if necessary.

Let it be required, for example, to find the second root of $192\,a^2\,b^3\,c$. The root is indicated thus, $\sqrt{192\,a^2\,b^3\,c}$. But $192\,a^2\,b^3\,c = 64\,.\,3\,a^2\,b^2\,b\,c$ or $64\,a^2\,b^2\,.\,3\,b\,c$. Now the first three factors, 64, a^2 and b^2, are second powers; we may, therefore, take the roots of these and place their product as a coëfficient to the expression indicating the root of $3\,b\,c$. We have then $\sqrt{192\,a^2\,b^3\,c} = 8\,a\,b\sqrt{3\,b\,c}$. It only remains now to approximate the root of $3\,b\,c$, the value of the letters supposed to be known, and multiply the result by $8\,a\,b$.

In separating an irrational quantity into factors for the purpose of simplifying, the learner has merely to find the greatest numerical factor that is an exact power, and the greatest exponent of each literal factor, not exceeding its given exponent, that is divisible by the number which marks the degree of the root.

1. Simplify $\sqrt{125\,a^3\,b^5}$.
2. Simplify $(80\,a\,b\,c^4)^{\frac{1}{2}}$.
3. Simplify $\sqrt[3]{108\,a^9\,b^6\,c^2}$.
4. Simplify $\sqrt{45\,a\,b^2\,c^2}$.
5. Simplify $\sqrt[3]{320\,a^3\,b - 64\,a^5\,b^3}$.

The greatest numerical factor in this quantity that is a third power is 64, and the greatest literal factor that is a third power is a^3. Hence, $320\,a^3\,b - 64\,a^5\,b^3 = 64\,a^3\,(5\,b - a^2\,b^3)$. Taking the root of $64\,a^3$, and indicating that of $5\,b - a^2\,b^3$, we have

$$\sqrt[3]{320\,a^3\,b - 64\,a^5\,b^3} = 4\,a\sqrt[3]{5\,b - a^2\,b^3}, \text{ or } 4\,a\,(5\,b - a^2\,b^3)^{\frac{1}{3}}$$

6. Simplify $\sqrt[3]{24\,a^4 - 8\,a^3\,b}$.
7. Simplify $(2\,a^3\,b^2 + a^5\,b\,c)^{\frac{1}{3}}$.

8. Simplify $\sqrt[3]{a^3 + a^3 b^3}$.

9. Simplify $\sqrt[4]{768\, a^6 - 256\, a^4}$.

10. Simplify $\sqrt{3456\, a^3 b - 1728}$.

If the quantity which is under the radical sign, or which is enclosed in a parenthesis with a fractional exponent, is a fraction, the expression may be simplified in the following manner, viz:

Multiply both terms of the fraction by such a quantity, as will render the denominator an exact power of the requisite degree, then take the roots of the denominator and of such factors of the numerator as are exact powers.

Remark. This preparation of the fraction is rarely advisable, except when the denominator is a monomial.

Thus, $\sqrt{\frac{3\, a^3}{8\, b}} = \sqrt{\frac{6\, a^3 b}{16\, b^2}} = \sqrt{\frac{a^2}{16\, b^2} \cdot 6\, a\, b} = \frac{a}{4\, b} \sqrt{6\, a\, b}$.

In like manner, $\sqrt[3]{\frac{4\, a^2}{9\, b}} = \sqrt[3]{\frac{12\, a^2 b^2}{27\, b^3}} =$

$$\sqrt[3]{\frac{1}{27\, b^3} \cdot 12\, a^2 b^2} = \frac{1}{3\, b} \sqrt[3]{12\, a^2 b^2}.$$

11. Simplify $\sqrt{\frac{64\, a}{27\, b}}$.

12. Simplify $\left(\frac{125\, a^4 b}{9\, c^2 d}\right)^{\frac{1}{2}}$.

13. Simplify $\sqrt[3]{\frac{a^2 b^2}{4\, c^3}}$.

14. Simplify $\left(\frac{3\, a}{4\, b}\right)^{\frac{1}{3}}$.

15. Simplify $\sqrt{\frac{3\, a^2 b}{16\, c}}$.

16. Simplify $\sqrt{\frac{12\, a^3 b - 8\, a^2 b^2}{27\, c^4 - 9\, c^2 d}}$.

17. Simplify $\left(\frac{1372\,a^4\,b - 343\,a^3\,b}{500\,m^4 - 250\,m^3\,n}\right)^{\frac{1}{2}}$.

18. Simplify $\sqrt[3]{\frac{320\,a^3\,b + 640\,a^5}{27\,c^4\,m + 54\,c^3\,m^2}}$.

Art. **123.** As we can extract the root of any factor and place it as a factor before the radical sign; so, if we would put under the sign any factor standing before it, we must raise that factor to a power of the same degree as the radical.

Thus $a\,b\sqrt{c} = \sqrt{a^2\,b^2\,c}$; and $\frac{3}{4}\sqrt[3]{a\,m} = \sqrt[3]{\frac{27\,a\,m}{64}}$.

Reduce the following quantities entirely to a radical form.

1. $3\,a\,x\sqrt{b}$.
2. $\frac{2}{3}\sqrt{5\,b\,c}$.
3. $\frac{2}{3}\sqrt[3]{\frac{a-b}{c+d}}$.
4. $(a+b)\sqrt{\frac{c}{3\,d}}$.
5. $2\,a\,(3\,b)^{\frac{1}{2}}$.
6. $5\,(x\,y)^{\frac{1}{3}}$.
7. $2\,a\,b\,(x+2\,y)^{\frac{1}{4}}$.
8. $\frac{3\,a}{4\,b}\,(m^2)^{\frac{1}{3}}$.

SECTION XLIV.

OPERATIONS ON IRRATIONAL QUANTITIES WITH FRACTIONAL EXPONENTS.

Art. **124.** In general, operations are performed on quantities with fractional exponents, in the same manner as if the exponents were entire.

Add $4\,a^{\frac{1}{2}}$ and $3\,a^{\frac{1}{2}}$.

The sum is $4\,a^{\frac{1}{2}} + 3\,a^{\frac{1}{2}} = 7\,a^{\frac{1}{2}}$.

Add $a\,b^{\frac{1}{2}}$ and $c\,b^{\frac{1}{2}}$.

The sum is $a\,b^{\frac{1}{2}} + c\,b^{\frac{1}{2}} = (a+c)\,b^{\frac{1}{2}}$.

From $9x^{\frac{1}{3}}$ subtract $4x^{\frac{1}{3}}$.

The difference is $9x^{\frac{1}{3}} - 4x^{\frac{1}{3}} = 5x^{\frac{1}{3}}$.

From $3ax^{\frac{1}{2}}y^{\frac{1}{3}}$ subtract $2bx^{\frac{1}{2}}y^{\frac{1}{3}}$.

The difference is $3ax^{\frac{1}{2}}y^{\frac{1}{3}} - 2bx^{\frac{1}{2}}y^{\frac{1}{3}} = (3a-2b)x^{\frac{1}{2}}y^{\frac{1}{3}}$.

Add $3 \cdot 8^{\frac{1}{2}}$ and $5 \cdot 18^{\frac{1}{2}}$.

The sum indicated is $3 \cdot 8^{\frac{1}{2}} + 5 \cdot 18^{\frac{1}{2}}$. But these terms may be simplified and the expression reduced. For, $3 \cdot 8^{\frac{1}{2}} = 3 \cdot 4^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 3 \cdot 2 \cdot 2^{\frac{1}{2}} = 6 \cdot 2^{\frac{1}{2}}$; and $5 \cdot 18^{\frac{1}{2}} = 5 \cdot 9^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 5 \cdot 3 \cdot 2^{\frac{1}{2}} = 15 \cdot 2^{\frac{1}{2}}$. Hence, $3 \cdot 8^{\frac{1}{2}} + 5 \cdot 18^{\frac{1}{2}} = 6 \cdot 2^{\frac{1}{2}} + 15 \cdot 2^{\frac{1}{2}} = 21\,(2)^{\frac{1}{2}}$.

In a similar manner, $(192b^3)^{\frac{1}{3}} + (24c^3)^{\frac{1}{3}} = 4b \cdot 3^{\frac{1}{3}} + 2c \cdot 3^{\frac{1}{3}} = (4b + 2c)\,3^{\frac{1}{3}}$.

Add $(\frac{8}{27})^{\frac{1}{2}}$ and $(\frac{1}{6})^{\frac{1}{2}}$.

The sum expressed is $(\frac{8}{27})^{\frac{1}{2}} + (\frac{1}{6})^{\frac{1}{2}}$. But $(\frac{8}{27})^{\frac{1}{2}} = (\frac{24}{81})^{\frac{1}{2}} = (\frac{4}{81} \cdot 6)^{\frac{1}{2}} = \frac{2}{9} \cdot 6^{\frac{1}{2}}$; and $(\frac{1}{6})^{\frac{1}{2}} = (\frac{6}{36})^{\frac{1}{2}} = (\frac{1}{36} \cdot 6)^{\frac{1}{2}} = \frac{1}{6} \cdot 6^{\frac{1}{2}}$.

Hence $(\frac{8}{27})^{\frac{1}{2}} + (\frac{1}{6})^{\frac{1}{2}} = \frac{2}{9} \cdot 6^{\frac{1}{2}} + \frac{1}{6} \cdot 6^{\frac{1}{2}} = \frac{4}{18} \cdot 6^{\frac{1}{2}} + \frac{3}{18} \cdot 6^{\frac{1}{2}} = \frac{7}{18}(6)^{\frac{1}{2}}$. The result therefore in its simplest form is $\frac{7}{18}(6)^{\frac{1}{2}}$.

We deduce therefore the following

RULE FOR THE ADDITION AND SUBTRACTION OF IRRATIONAL QUANTITIES.

Express the addition or subtraction as usual by signs, simplify the terms if possible, and reduce similar terms.

Remark. Irrational quantities, indicated by means of fractional exponents, are similar, when the factors having fractional exponents are alike in all, and have severally the same exponents.

Multiply $a^{\frac{1}{5}}$ by $a^{\frac{2}{5}}$.

This is performed by adding the exponents. Thus, $a^{\frac{1}{5}} \cdot a^{\frac{2}{5}} = a^{\frac{1}{5}+\frac{2}{5}} = a^{\frac{3}{5}}$.

Multiply $3\,a^{\frac{2}{7}}\,b^{\frac{1}{5}}$ by $5\,a^{\frac{3}{7}}\,b^{\frac{3}{5}}$.

The product is $15\,a^{\frac{2}{7}+\frac{3}{7}}\,b^{\frac{1}{5}+\frac{3}{5}} = 15\,a^{\frac{5}{7}}\,b^{\frac{4}{5}}$.

Multiply $2\,a^{\frac{1}{3}}$ by $3\,a^{\frac{2}{5}}$.

Reducing the exponents to a common denominator, we have $2\,a^{\frac{1}{3}} = 2\,a^{\frac{5}{15}}$, and $3\,a^{\frac{2}{5}} = 3\,a^{\frac{6}{15}}$; therefore, $2\,a^{\frac{1}{3}} \,.\, 3\,a^{\frac{2}{5}} = 2\,a^{\frac{5}{15}} \,.\, 3\,a^{\frac{6}{15}} = 6\,a^{\frac{11}{15}}$.

Divide $a^{\frac{4}{5}}$ by $a^{\frac{1}{5}}$.

This is performed by subtracting the exponent of the latter from that of the former. Thus, $\dfrac{a^{\frac{4}{5}}}{a^{\frac{1}{5}}} = a^{\frac{4}{5}-\frac{1}{5}} = a^{\frac{3}{5}}$.

Divide $15\,a^{\frac{6}{7}}\,b^{\frac{5}{3}}$ by $3\,a^{\frac{4}{7}}\,b^{\frac{4}{3}}$.

The quotient is $\dfrac{15\,a^{\frac{6}{7}}\,b^{\frac{5}{3}}}{3\,a^{\frac{4}{7}}\,b^{\frac{4}{3}}} = 5\,a^{\frac{2}{7}}\,b^{\frac{1}{3}}$.

Divide $3\,a^{\frac{1}{2}}\,b^{\frac{2}{5}}$ by $5\,a^{\frac{1}{3}}\,b^{\frac{1}{4}}$.

Reducing the exponents of the similar factors to a common denominator, we have $\dfrac{3\,a^{\frac{1}{2}}\,b^{\frac{2}{5}}}{5\,a^{\frac{1}{3}}\,b^{\frac{1}{4}}} = \dfrac{3\,a^{\frac{3}{6}}\,b^{\frac{8}{20}}}{5\,a^{\frac{2}{6}}\,b^{\frac{5}{20}}} = \dfrac{3\,a^{\frac{1}{6}}\,b^{\frac{3}{20}}}{5}$.

Find the second power of $3\,a^{\frac{2}{9}}$.

This is performed by raising the coëfficient to the required power, and multiplying the exponent by the index of the power. Thus, $\left(3\,a^{\frac{2}{9}}\right)^2 = 9\,a^{\frac{4}{9}}$.

In like manner, $\left(3\,a^{\frac{5}{12}}\,b^{\frac{1}{5}}\right)^4 = 81\,a^{\frac{5}{3}}\,b^{\frac{4}{5}}$.

Conversely, the root of an irrational quantity is found, by taking or expressing the root of the numerical coëfficient, and dividing the exponents of the other factors by the number which marks the degree of the root.

Thus, the second root of $a^{\frac{4}{9}}$ is $a^{\frac{2}{9}}$, and the fourth root of $a^{\frac{5}{3}}$

is $a^{\frac{5}{12}}$. Also the third root of $27\,a^{\frac{3}{4}}\,b^{\frac{1}{5}}$ is $3\,a^{\frac{1}{4}}\,b^{\frac{1}{15}}$; and the fifth root of $7\,a^{\frac{1}{2}}\,b^{\frac{1}{3}}$ is $7^{\frac{1}{5}}\,a^{\frac{1}{10}}\,b^{\frac{1}{15}}$.

From what precedes, we see that the following operations, viz: *multiplication, division, finding powers, and extracting roots, are performed upon quantities with fractional exponents, in the same manner as if the exponents were whole numbers.*

Art. **125**. In the multiplication of irrational quantities, we assumed that their fractional exponents might be reduced to equivalent ones having a common denominator, without changing the value of the quantities. This, however, may be easily proved.

For, multiplying the numerator of the exponent of any quantity, raises that quantity to a power, and multiplying the denominator, divides the exponent, and therefore extracts the root. Consequently, when both terms of a fractional exponent are multiplied by the same number, which is done in reducing to a common denominator, the quantity to which the exponent belongs, is raised to a power of a certain degree, and then the root of the result is extracted to the same degree; or the reverse. The value of the quantity, therefore, remains unchanged.

Accordingly, the exponents of all the factors in any product, may be reduced to fractions having a common denominator. Thus, $2\,a^{\frac{1}{3}}\,b^{\frac{1}{2}} = 2^{\frac{6}{6}}\,a^{\frac{2}{6}}\,b^{\frac{3}{6}} = 64^{\frac{1}{6}}\,a^{\frac{2}{6}}\,b^{\frac{3}{6}} = (64\,a^2\,b^3)^{\frac{1}{6}}$.

Moreover, it is manifest that the fractional exponents may be reduced to decimals, and that the value of the result will be either exactly or approximately the same as that of the given quantity.

For example, $a^{\frac{1}{4}} = a^{0 \cdot 25}$, and $a^{\frac{7}{8}} = a^{0 \cdot 875}$. If it were required then to multiply $a^{\frac{1}{4}}$ by $a^{\frac{7}{8}}$, we should have $a^{\frac{1}{4}} \,.\, a^{\frac{7}{8}} = a^{0 \cdot 25} \,.\, a^{0 \cdot 875}$ $= a^{0 \cdot 25 + 0 \cdot 875} = a^{1 \cdot 125}$.

1. Add $(27\,a^4\,x)^{\frac{1}{2}}$ and $(3\,a^4\,x)^{\frac{1}{2}}$.

2. Add $(128)^{\frac{1}{2}}$ and $(72)^{\frac{1}{2}}$.

3. Add $(135)^{\frac{1}{3}}$ and $(40)^{\frac{1}{3}}$.

4. Add $(12)^{\frac{1}{2}}$, $2(27)^{\frac{1}{2}}$ and $3(75)^{\frac{1}{2}}$.

5. Add $2(8)^{\frac{1}{2}} - 7(18)^{\frac{1}{2}}$ and $5(72)^{\frac{1}{2}} - (50)^{\frac{1}{2}}$.

6. Add $7(54)^{\frac{1}{3}}$, $3(16)^{\frac{1}{3}}$ and $(2)^{\frac{1}{3}} - 5(128)^{\frac{1}{3}}$.

7. Add $(4a^2b)^{\frac{1}{2}}$, $3ab^{\frac{1}{2}}$, $(27a^3b)^{\frac{1}{3}}$ and $(125a^3b)^{\frac{1}{3}}$.

8. Add $3\left(\frac{5}{27}\right)^{\frac{1}{2}}$ and $4\left(\frac{3}{5}\right)^{\frac{1}{2}}$.

9. Add $3\left(\dfrac{2a^2}{5}\right)^{\frac{1}{2}}$ and $2\left(\dfrac{a^2}{10}\right)^{\frac{1}{2}}$.

10. Add $(9ab)^{\frac{1}{2}}$, $(c^2ab)^{\frac{1}{2}}$, $\left(\dfrac{49ab}{4}\right)^{\frac{1}{2}}$ and $(xy)^{\frac{1}{2}}$.

11. From $(18)^{\frac{1}{2}}$ subtract $8^{\frac{1}{2}}$

12. From $(108ax^2)^{\frac{1}{2}}$ subtract $(48ax^2)^{\frac{1}{2}}$.

13. From $(432a^3b)^{\frac{1}{3}}$ subtract $(16a^3b)^{\frac{1}{3}}$.

14. From $(192a^4b^2)^{\frac{1}{3}}$ subtract $(24a^4b^2)^{\frac{1}{3}}$.

15. From $\frac{3}{4}\left(\frac{2}{3}\right)^{\frac{1}{2}}$ subtract $\frac{2}{5}\left(\frac{1}{6}\right)^{\frac{1}{2}}$.

16. From $5(20)^{\frac{1}{2}}$ subtract $3(45)^{\frac{1}{2}}$.

17. From $(16ab)^{\frac{1}{2}} - (343m)^{\frac{1}{3}}$ subtract $(9ab)^{\frac{1}{2}} - (1000c^3m)^{\frac{1}{3}}$.

18. From $\left(\dfrac{49ab}{192}\right)^{\frac{1}{2}} - \left(\dfrac{8m}{135}\right)^{\frac{1}{3}}$ subtract $\left(\dfrac{4ab}{27}\right)^{\frac{1}{2}} - \left(\dfrac{27m}{320}\right)^{\frac{1}{3}}$.

19. Multiply $7a^{\frac{1}{4}}b^{\frac{1}{2}}$ by $3a^{\frac{3}{4}}b^{\frac{3}{2}}$.

20. Multiply $2abc$ by $5a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$.

21. Multiply $ma^{\frac{3}{5}}c^{\frac{1}{4}}$ by $3ma^{\frac{3}{8}}c^{\frac{2}{5}}$.

22. Multiply $25x^{\frac{1}{2}}y$ by $3x^{\frac{1}{3}}y^{\frac{1}{6}}$.

23. Multiply $10(108)^{\frac{1}{3}}$ by $5(4)^{\frac{1}{3}}$.

The product is $50(108)^{\frac{1}{3}}(4)^{\frac{1}{3}} = 50(432)^{\frac{1}{3}} = 50(216 \,.\, 2)^{\frac{1}{3}}$ $= 50 \,.\, 6 \,.\, 2^{\frac{1}{3}} = 300(2)^{\frac{1}{3}}$.

24. Multiply $5 \,.\, 5^{\frac{1}{2}}$ by $3 \,.\, 8^{\frac{1}{2}}$, and simplify.

25. Multiply $2 \,.\, 3^{\frac{1}{2}}$ by $3 \,.\, 4^{\frac{1}{3}}$.

The product is $6 \,.\, 3^{\frac{1}{2}} \,.\, 4^{\frac{1}{3}} = 6 \,.\, 3^{\frac{3}{6}} \,.\, 4^{\frac{2}{6}} = 6 \,.\, 27^{\frac{1}{6}} \,.\, 16^{\frac{1}{6}} = 6\,(432)^{\frac{1}{6}}$.

26. What is the product of 4, $2\,(3)^{\frac{1}{6}}$ and $72^{\frac{1}{6}}$?

27. What is the product of $5\,(3)^{\frac{1}{2}}$, $7\,(\frac{8}{3})^{\frac{1}{2}}$ and $2^{\frac{1}{2}}$?

28. Multiply $a^{\frac{1}{m}}$ by $a^{\frac{1}{n}}$.

29. Multiply $3\,a^{\frac{m}{n}}$ by $5\,a^{\frac{p}{q}}$.

30. Multiply together $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$.

31. Multiply $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$.

32. Multiply $3\,(c-d)^{\frac{1}{7}}$ by $4\,a\,(c-d)^{\frac{7}{8}}$.

33. Multiply $4\,a^2\,b\,(x-y)^{\frac{1}{6}}$ by $3\,(a+b)^{\frac{1}{3}}\,(x-y)^{\frac{6}{5}}$.

34. Multiply $5\,(m+n)^{\frac{1}{2}}\,(c-d)^{\frac{1}{5}}$ by $7\,(m+n)^{\frac{1}{3}}\,(c-d)^{\frac{1}{3}}$.

35. Multiply $3+5^{\frac{1}{2}}$ by $3-5^{\frac{1}{2}}$.

36. Multiply $7+2\,(6)^{\frac{1}{2}}$ by $9-5\,(6)^{\frac{1}{2}}$.

37. Multiply $9+2\,(10)^{\frac{1}{2}}$ by $9-2\,(10)^{\frac{1}{2}}$.

38. Divide $a^{\frac{5}{2}}$ by $a^{\frac{1}{2}}$.

39. Divide $a\,b^{\frac{1}{2}}\,c$ by $a^{\frac{1}{3}}\,b^{\frac{1}{6}}\,c^{\frac{1}{5}}$.

40. Divide $6\,a^{\frac{9}{10}}\,b^{\frac{5}{2}}\,c^3$ by $3\,a^{\frac{7}{8}}\,b\,c^2$.

41. Divide $3\,a^{\frac{2}{3}}\,b^{\frac{5}{6}}\,c^2$ by $4\,a^{\frac{1}{2}}\,b^{\frac{1}{2}}\,c^{\frac{1}{3}}$.

42. Divide $10\,(108)^{\frac{1}{3}}$ by $5\,(4)^{\frac{1}{3}}$.

43. Divide $10\,(27)^{\frac{1}{2}}$ by $2\,(3)^{\frac{1}{2}}$.

44. Divide $8\,(512)^{\frac{1}{3}}$ by $4\,(2)^{\frac{1}{2}}$.

45. Divide $a^{\frac{1}{m}}$ by $a^{\frac{1}{n}}$.

46. Divide $3\,a^{\frac{1}{m}}\,b^{\frac{1}{n}}$ by $4\,a^{\frac{1}{p}}\,b^{\frac{1}{q}}$.

47. Divide $\frac{33}{7}\,(\frac{2}{3})^{\frac{1}{2}}$ by $\frac{12}{5}\,(\frac{3}{2})^{\frac{1}{2}}$.

48. Divide $(a-b)^{\frac{7}{8}}$ by $(a-b)^{\frac{3}{4}}$.

49. Divide $12\,(x-y)^{\frac{1}{2}}$ by $4\,a\,(x-y)^{\frac{1}{3}}$.

50. Divide $13\,(a+b)^{\frac{7}{8}}\,(c-d)^{\frac{1}{3}}$ by $39\,m\,(a+b)^{\frac{3}{4}}\,(c-d)^{\frac{1}{5}}$.

51. Find the 2d power of $2\,a^{\frac{1}{3}}\,b^{\frac{1}{4}}$.

52. Find the 3d power of $5\,a^2\,b^{\frac{1}{5}}\,c^{\frac{1}{3}}$.

53. Find the 3d power of $\dfrac{2\,a^{\frac{1}{2}}\,b^{\frac{2}{3}}}{3\,x^{\frac{2}{3}}\,y}$.

54. Find the 4th power of $\frac{1}{2}\,.\,3^{\frac{1}{3}}$.

55. Find the 5th power of $\dfrac{2\,.\,3^{\frac{2}{5}}\,a^{\frac{1}{4}}\,b\,c^2}{m^{\frac{1}{5}}}$.

56. Find the 3d power of $(a+b)^{\frac{1}{5}}$.

57. Find the 3d power of $3\,(x-y)^{\frac{7}{9}}$.

58. Find the 5th power of $2\,(x^2-y^2)^{\frac{7}{8}}\,(b-c)^{\frac{3}{4}}$.

59. Find the *m*th power of $a^{\frac{1}{6}}\,b^{\frac{3}{7}}\,c$.

60. Find the *m*th power of $a^{\frac{1}{n}}\,b^{\frac{1}{p}}\,c\,d$.

61. Find the 3d power of $\dfrac{2\,(a+b)^{\frac{1}{8}}}{3\,(c-d)^{\frac{1}{5}}}$.

62. Find the 2d power of $\dfrac{2\,(x+y)^{\frac{1}{5}}\,(c-d)^{\frac{1}{6}}}{5\,(m+n)^{\frac{1}{4}}\,(x-y)^{\frac{3}{4}}}$.

63. Extract the 2d root of $a^{\frac{2}{7}}\,b^{\frac{4}{5}}$.

64. Extract the 3d root of $27\,a^{\frac{3}{5}}\,b^{\frac{6}{7}}$.

65. Extract the 3d root of $2\,a^{\frac{1}{2}}\,b^{\frac{1}{3}}$.

66. Extract the 4th root of $3\,a^{\frac{3}{7}}\,b^2\,c$.

67. Extract the *m*th root of $10\,a^{\frac{3}{4}}\,x\,y^{\frac{1}{3}}$.

68. Extract the *m*th root of $6\,a^{\frac{p}{q}}\,b^{\frac{r}{s}}$.

69. Extract the 2d root of $(a+b)^{\frac{2}{5}}$.

70. Extract the 2d root of $16\,(a-b)^{\frac{4}{5}}\,(c-d)^{\frac{1}{3}}$.

71. Extract the 3d root of $8\,a^3\,(m-n)^{\frac{2}{3}}\,(c-d)^{\frac{1}{2}}$.

72. Extract the 3d root of $\dfrac{3\,(x-y)^{\frac{2}{7}}\,(a^2+d)^2}{5\,b\,(m+n)^{\frac{2}{5}}}$.

SECTION XLV.

OPERATIONS UPON IRRATIONAL QUANTITIES WITH RADICAL SIGNS.

Art. **126**. Although radical signs may be wholly dispensed with, and fractional exponents used instead of them, yet, as these signs occur in almost all mathematical treatises, and are sometimes very convenient, we shall show how to perform the various operations on quantities affected with them.

Irrational quantities affected with radical signs, are commonly called *radical quantities,* the mode of simplifying which has already been shown.

The addition and subtraction of radical quantities are, it is manifest, performed in the same manner as when fractional exponents are used. We observe, however, that radical quantities are said to be similar, when they are of the same degree and have the quantities under the sign in all respects similar.

Let it be required to add together $\sqrt[3]{192}$ and $\sqrt[3]{24}$. The sum expressed is $\sqrt[3]{192}+\sqrt[3]{24}$. But $\sqrt[3]{192}=\sqrt[3]{64\,.\,3}=4\sqrt[3]{3}$, and $\sqrt[3]{24}=\sqrt[3]{8\,.\,3}=2\sqrt[3]{3}$; hence, $\sqrt[3]{192}+\sqrt[3]{24}=4\sqrt[3]{3}+2\sqrt[3]{3}=6\sqrt[3]{3}$.

In like manner, the sum of $m\sqrt{a^3\,b^4\,c}$ and $b^2\,n\sqrt{a^3\,c}$ is $(a\,b^2\,m+a\,b^2\,n)\sqrt{a\,c}$, or $a\,b^2\,(m+n)\sqrt{a\,c}$.

Subtract $\sqrt[3]{108}$ from $9\sqrt[3]{4}$. The difference expressed is $9\sqrt[3]{4}-\sqrt[3]{108}$, which simplified becomes $9\sqrt[3]{4}-3\sqrt[3]{4}=6\sqrt[3]{4}$.

In like manner, $\sqrt[4]{a^4 b}$ subtracted from $3m\sqrt[4]{c^4 b}$, leaves $3mc\sqrt[4]{b}-a\sqrt[4]{b}=(3mc-a)\sqrt[4]{b}$.

Art. **127**. Rules for other operations on radicals, may be easily deduced from the modes given in the preceding section, for performing corresponding operations on irrational quantities with fractional exponents.

The exponents of the quantities under the radical sign and the index over that sign, may both be multiplied or divided by the same number without affecting the value of the expression.

For example, $\sqrt{a^3}$ or $\sqrt[2]{a^3}=a^{\frac{3}{2}}=a^{\frac{15}{10}}=\sqrt[10]{a^{15}}$, which might have been obtained by multiplying the 2 and 3 of the expression $\sqrt[2]{a^3}$ both by 5.

In like manner, $\sqrt[3]{a^2 b}=\sqrt[9]{a^6 b^3}$.

Again, $\sqrt[10]{a^{15}}=a^{\frac{15}{10}}=a^{\frac{3}{2}}=\sqrt[2]{a^3}$ or $\sqrt{a^3}$, which is obtained by dividing the 10 and 15 of the expression $\sqrt[10]{a^{15}}$ both by 5.

In a similar manner, $\sqrt[9]{a^6 b^3}=\sqrt[3]{a^2 b}$.

Art. **128**. Hence, two or more radical expressions may be made to have the same index over the sign. Thus, $\sqrt{a^3}$ and $\sqrt[3]{b^2}$ are respectively the same as $\sqrt[6]{a^9}$ and $\sqrt[6]{b^4}$; also $\sqrt[4]{a^3 b}$ and $\sqrt[6]{x y^3}$ are respectively the same as $\sqrt[12]{a^9 b^3}$ and $\sqrt[12]{x^2 y^6}$.

This process is evidently the same as reducing fractional exponents to a common denominator, the indices over the sign being considered as denominators, and the exponents of the quantities under the sign, as numerators.

The common index will therefore be the product of all the indices over the sign, or their least common multiple.

Art. **129**. Multiply $\sqrt{a}$ by $\sqrt{b}$.

The product is $\sqrt{ab}$; for $\sqrt{a} = a^{\frac{1}{2}}$, and $\sqrt{b} = b^{\frac{1}{2}}$; therefore, $\sqrt{a} \,.\, \sqrt{b} = a^{\frac{1}{2}}\, b^{\frac{1}{2}} = (a\, b)^{\frac{1}{2}} = \sqrt{a\, b}$.

Multiply $2\sqrt[3]{a^2}$ by $3\sqrt[5]{b^3}$.

We first render the indices over the sign alike; we then have $2\sqrt[3]{a^2} \,.\, 3\sqrt[5]{b^3} = 2\sqrt[15]{a^{10}} \,.\, 3\sqrt[15]{b^9} = 6\, a^{\frac{10}{15}}\, b^{\frac{9}{15}} = 6\, (a^{10}\, b^9)^{\frac{1}{15}}$ $= 6\sqrt[15]{a^{10}\, b^9}$.

Divide $6\sqrt{a\, b}$ by $3\sqrt{a}$. The quotient expressed is $\dfrac{6\sqrt{a\, b}}{3\sqrt{a}}$.

But $6\sqrt{a\, b} = 6\, a^{\frac{1}{2}}\, b^{\frac{1}{2}}$, and $3\sqrt{a} = 3\, a^{\frac{1}{2}}$; therefore $\dfrac{6\sqrt{a\, b}}{3\sqrt{a}} =$ $\dfrac{6\, a^{\frac{1}{2}}\, b^{\frac{1}{2}}}{3\, a^{\frac{1}{2}}} = 2\, b^{\frac{1}{2}} = 2\sqrt{b}$.

Divide $3\sqrt{a}$ by $5\sqrt[3]{b}$ Making the indices alike and then dividing, we have $\dfrac{3\sqrt{a}}{5\sqrt[3]{b}} = \dfrac{3\sqrt[6]{a^3}}{5\sqrt[6]{b^2}} = \dfrac{3}{5} \,.\, \dfrac{a^{\frac{3}{6}}}{b^{\frac{2}{6}}} = \dfrac{3}{5}\left(\dfrac{a^3}{b^2}\right)^{\frac{1}{6}} =$ $\dfrac{3}{5}\sqrt[6]{\dfrac{a^3}{b^2}}$.

Hence, we have the following

RULE FOR THE MULTIPLICATION AND DIVISION OF RADICALS.

Make the indices over the radical sign alike, if they are not so; then multiply or divide one coëfficient by the other; also take the product or quotient of the quantities under the radical sign, placing the latter result under the common sign, which is to be preceded by the product or quotient of the coëfficients.

Art. **130**. Find the fifth power of $2\sqrt[3]{a^2\, b}$.

Since $2\sqrt[3]{a^2\, b} = 2\, a^{\frac{2}{3}}\, b^{\frac{1}{3}}$, we have $(2\sqrt[3]{a^2\, b})^5 = (2\, a^{\frac{2}{3}}\, b^{\frac{1}{3}})^5$ $= 2^5 \,.\, a^{\frac{2}{3} \times 5}\, b^{\frac{1}{3} \times 5} = 32\, a^{\frac{10}{3}}\, b^{\frac{5}{3}} = 32\, (a^{10}\, b^5)^{\frac{1}{3}} = 32\sqrt[3]{a^{10}\, b^5}$.

This result might have been obtained from $2\sqrt[3]{a^2 b}$ by raising the coëfficient 2 to the fifth power, and multiplying the exponents of the quantities under the radical sign by 5.

Raise $3\sqrt[9]{a^2}$ to the third power.

Since $3\sqrt[9]{a^2} = 3a^{\frac{2}{9}}$, we have $(3\sqrt[9]{a^2})^3 = (3a^{\frac{2}{9}})^3 = 3^3 \times a^{\frac{2}{9}\times 3} = 27a^{\frac{2}{3}} = 27\sqrt[3]{a^2}$. This result is obtained from $3\sqrt[9]{a^2}$, by raising the coëfficient 3 to the third power, and dividing the index 9 by 3. Hence we have the following

RULE FOR RAISING A RADICAL TO ANY POWER.

Raise the coëfficient to the power required, and either raise the quantity under the radical sign to the same power, or divide the index over it by the number expressing the degree of the power.

Art. **131**. Since extracting a root is the reverse of finding a power, we have the following

RULE FOR EXTRACTING ANY ROOT OF A RADICAL.

Extract the root of the coëfficient, and either extract the root of the quantity under the radical sign, or multiply the index over it by the number expressing the degree of the root.

Thus, the third root of $8\sqrt[7]{a^9}$ is $2\sqrt[7]{a^3}$; and the fourth root of $81\sqrt[3]{a^5}$ is $\pm 3\sqrt[12]{a^5}$.

The fifth root of $4\sqrt[3]{a^2}$ is $\sqrt[5]{4} \,.\, \sqrt[15]{a^2} = \sqrt[15]{4^3} \,.\, \sqrt[15]{a^2} = \sqrt[15]{4^3 \,.\, a^2} = \sqrt[15]{64\,a^2}$.

Art. **132**. The division of irrational quantities often gives rise to fractions, whose numerators and denominators are both irrational. In such a case, it is often desirable to convert the fraction into another equivalent to it, but of a simpler form. This may be accomplished by multiplying both terms of the fraction, by any quantity, which will render one of them rational

Thus, if both terms of $\frac{\sqrt{a}}{\sqrt{x}}$ or $\sqrt{\frac{a}{x}}$ be multiplied by $\sqrt{a}$, we have $\frac{a}{\sqrt{ax}}$, or if both be multiplied by $\sqrt{x}$, we have $\frac{\sqrt{ax}}{x}$. In like manner, multiplying both terms of the fraction $\frac{\sqrt{b}}{\sqrt[3]{a+x}}$ by $\sqrt[3]{(a+x)^2}$, gives $\frac{\sqrt{b}\,.\,\sqrt[3]{(a+x)^2}}{\sqrt[3]{(a+x)^3}} = \frac{\sqrt{b}\,.\,\sqrt[3]{(a+x)^2}}{a+x} = \frac{\sqrt[6]{b^3}\,.\,\sqrt[6]{(a+x)^4}}{a+x} = \frac{\sqrt[6]{b^3(a+x)^4}}{a+x}$; also, $\frac{a^{\frac{1}{2}}}{x^{\frac{1}{n}}} = \frac{a^{\frac{1}{2}}\,x^{\frac{n-1}{n}}}{x^{\frac{1}{n}}\,x^{\frac{n-1}{n}}} = \frac{a^{\frac{1}{2}}\,x^{\frac{n-1}{n}}}{x}$.

Since the product of the sum and difference of two quantities is the difference of their second powers, Art. **33**, if we would render the denominator of $\frac{\sqrt{2}}{3-\sqrt{2}}$ rational, we multiply both terms of the fraction by $3+\sqrt{2}$. We have then $\frac{\sqrt{2}}{3-\sqrt{2}} = \frac{\sqrt{2}\,(3+\sqrt{2})}{(3-\sqrt{2})\,(3+\sqrt{2})} = \frac{3\sqrt{2}+2}{3^2-(\sqrt{2})^2} = \frac{3\sqrt{2}+2}{9-2} = \frac{3\sqrt{2}+2}{7}$. Also, multiplying both terms of the fraction $\frac{a-\sqrt{b}}{a+\sqrt{b}}$ by $a-\sqrt{b}$, we have $\frac{a^2-2a\sqrt{b}+b}{a^2-b}$.

To render the denominator of $\frac{\sqrt{3}}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$ rational, first multiply numerator and denominator by $\sqrt{10}+\sqrt{2}+\sqrt{3}$,

which gives $\dfrac{\sqrt{30}+\sqrt{6}+3}{5-2\sqrt{6}}$; then multiply both terms of this last by $5+2\sqrt{6}$, which gives $\dfrac{5\sqrt{30}+2\sqrt{180}+11\sqrt{6}+27}{25-24}$

$= 5\sqrt{30}+12\sqrt{5}+11\sqrt{6}+27.$

Simplifications of this kind may be made in fractions involving radicals of other degrees than the second; but, except when the quantity to be rendered rational is a monomial, the process becomes so complicated as to be inconsistent with the design of this treatise.

Remark. In the following questions, let the learner simplify his results, when it can be done.

1. Add $\sqrt{8}$ and $\sqrt{50}$.
2. Add $\sqrt{16b}$ and $\sqrt{4b}$.
3. Add $\sqrt{36a^2y}$ and $\sqrt{25y}$.
4. Add $\sqrt[3]{500}$ and $\sqrt[3]{108}$.
5. Add $4\sqrt{147}$ and $3\sqrt{75}$.
6. Add $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.
7. Add $9\sqrt{243}$ and $10\sqrt{363}$.
8. Add $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.
9. Add $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.
10. Add $\sqrt{12}+2\sqrt{27}$ and $3\sqrt{75}-9\sqrt{48}$.
11. Add $7\sqrt[3]{54}+3\sqrt[3]{16}$ and $\sqrt[3]{2}-5\sqrt[3]{128}$.
12. Add $\sqrt[3]{81}-2\sqrt[3]{24}$ and $\sqrt{28}+2\sqrt{63}$.
13. Add $\sqrt{18a^5b^3}$ and $\sqrt{50a^3b^3}$.
14. Add $\sqrt{45c^3}-\sqrt{80c^3}$ and $\sqrt{5a^2c}$.
15. Add $\sqrt{\dfrac{a^4c}{b^3}}$ and $\sqrt{\dfrac{a^2c^3}{bd^2}}-\sqrt{\dfrac{a^2cd^2}{be^2}}$.

16. From $\sqrt{50}$ subtract $\sqrt{8}$.

17. From $\sqrt{448}$ subtract $\sqrt{112}$.

18. From $\sqrt[3]{192}$ subtract $\sqrt[3]{24}$.

19. From $5\sqrt{20}$ subtract $3\sqrt{45}$.

20. From $\sqrt[3]{320}$ subtract $\sqrt[3]{40}$.

21. From $\sqrt{\frac{3}{5}}$ subtract $\sqrt{\frac{5}{27}}$.

22. From $\sqrt{8}$ subtract $2\sqrt{\frac{1}{2}}$.

23. From $\sqrt[3]{72}$ subtract $3\sqrt[3]{\frac{1}{3}}$.

24. From $\sqrt{80\, a^4\, x}$ subtract $\sqrt{20\, a^2\, x^3}$.

25. From $8\sqrt[3]{a^3\, b}$ subtract $2\sqrt[3]{a^6\, b}$.

26. From $\sqrt[3]{256}$ subtract $\sqrt{32}$.

27. From $\sqrt[3]{\frac{25}{9}}$ subtract $\sqrt[3]{\frac{3}{5}}$.

28. From $7\sqrt[3]{54} + 3\sqrt[3]{16}$ subtract $5\sqrt[3]{128} - \sqrt[3]{2}$.

29. From $\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}}$ subtract $\sqrt{\frac{3}{8}} - \sqrt{\frac{2}{3}}$.

30. Multiply $3\sqrt{2}$ by $2\sqrt{2}$.

31. Multiply $\sqrt{2}$ by $\sqrt{8}$.

32. Multiply $\sqrt[3]{2}$ by $\sqrt[3]{4}$.

33. Multiply $\sqrt{a}$ by $\sqrt{b}$.

34. Multiply $2\sqrt{a\, b}$ by $3\sqrt{a\, c}$.

35. Multiply $5\sqrt[3]{a^2\, c^2}$ by $a\sqrt{a\, c}$.

36. Multiply $5\sqrt{5}$ by $3\sqrt{8}$.

37. Multiply $2\sqrt{3}$ by $3\sqrt[3]{4}$.

38. Multiply $2\, a + 3\sqrt{b}$ by $2\, a - 3\sqrt{b}$.

39. Multiply $7 + 3\sqrt{12}$ by $3 + 4\sqrt[3]{2}$.

40. Multiply $3 + \sqrt{5}$ by $2 - \sqrt{5}$.

41. Multiply $\sqrt{2}+\sqrt{3}$ by $2\sqrt{2}-\sqrt{3}$.

42. Multiply $b\sqrt[3]{a^2}$ by $c\sqrt[5]{a}$.

43. Multiply $\sqrt[3]{\frac{1}{2}}$ by $\sqrt[3]{\frac{1}{4}}$.

44. Multiply $\frac{5}{2}\sqrt{\frac{3}{8}}$ by $\frac{9}{10}\sqrt{\frac{2}{5}}$.

45. Multiply together $\sqrt{2}$, $\sqrt{6}$ and $\sqrt{12}$.

46. Multiply together $2\sqrt{3}$, $3\sqrt[3]{4}$ and $6\sqrt[4]{8}$.

47. Multiply together $3\sqrt{ab}$, $4\sqrt[5]{ab}$ and $2\sqrt[3]{m}$.

48. Multiply together $\frac{1}{2}\sqrt{\frac{1}{2}}$, $\frac{1}{3}\sqrt[3]{7}$ and $\sqrt[7]{b}$.

49. Divide $6\sqrt{a}$ by $3\sqrt{a}$.

50. Divide $8\sqrt{ab}$ by $2\sqrt{a}$.

51. Divide $3\sqrt{xy}$ by $5\sqrt{x^2y^2}$.

52. Divide $8\sqrt{108}$ by $2\sqrt{6}$.

53. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.

54. Divide $\sqrt{7}$ by $\sqrt[3]{7}$.

55. Divide $3\sqrt[3]{ab}$ by $2\sqrt[5]{ab}$.

56. Divide $a\sqrt{xy}$ by $b\sqrt[3]{4c}$.

57. Divide $a+\sqrt{b}$ by $a-\sqrt{b}$.

Remark. In this and the two following examples, first represent the division, and then simplify by rendering the denominators rational.

58. Divide $4+\sqrt{2}$ by $3+\sqrt{2}$.

59. Divide $\sqrt{3}$ by $3+\sqrt{3}$.

60. Find the 2d power of $\sqrt[3]{a}$.

61. Find the 2d power of $5\sqrt[5]{b^2}$.

62. Find the 2d power of $3\sqrt[4]{ab}$.

63. Find the 3d power of $4\sqrt[6]{ax}$.

64. Find the 4th power of $a\sqrt[8]{b^3 c^2}$.

65. Find the 5th power of $m^2\sqrt[3]{a^2 b^2}$.

66. Find the *m*th power of $x\,y\sqrt{a\,b}$.

67. Find the *m*th power of $\sqrt[mn]{x\,y}$.

68. Find the 3d power of $\frac{1}{2}\sqrt{3}$.

69. Find the 4th power of $\frac{1}{6}\sqrt{6\,a}$.

70. Find the 3d power of $\frac{1}{2}\sqrt[3]{24}$.

71. Find the 5th power of $\sqrt{(a+b)^2}$.

72. Find the 2d root of $4\sqrt[3]{a^2 b^2}$.

73. Find the 3d root of $27\sqrt[6]{a\,b}$.

74. Find the 3d root of $64\sqrt[3]{a\,m}$.

75. Find the 4th root of $16\sqrt[5]{a^4 b^4 c^8}$.

76. Find the 3d root of $\frac{1}{8}\sqrt{a^3 b^3}$.

77. Find the 5th root of $\sqrt{a\,m}$.

78. Find the *m*th root of $\sqrt[3]{(a+b)^2}$.

79. Find the *m*th root of $\sqrt[3]{(a-b)^{2m}}$.

80. Find the *m*th root of $\sqrt[n]{x+y}$.

81. Find the 3d root of $3\sqrt{a+b}$.

SECTION XLVI.

NEGATIVE EXPONENTS.

Art. **133**. We have already seen, that, to divide one power of a quantity by another power of the same quantity, we must subtract the exponent of the divisor from that of the dividend. Thus, a^7 divided by a^3 gives $a^{7-3}=a^4$. We have also seen,

that when the dividend and divisor are alike, the quotient has zero for an exponent, and is equal to unity. Thus, $\frac{a^3}{a^3} = a^0 = 1$, and $\frac{(a+b)^2}{(a+b)^2} = (a+b)^0 = 1$.

If, however, the exponent of the divisor is greater than that of the dividend, the quotient will have a negative exponent. Thus, $\frac{a^3}{a^7} = a^{3-7} = a^{-4}$.

In order to understand the signification of negative exponents, let us take any fraction as $\frac{a}{a^2}$, which has different powers of the same letter for its two terms, but in which the exponent of the denominator exceeds by 1 that of the numerator. This fraction, reduced to its lowest terms, becomes $\frac{1}{a}$; but if the division represented, be performed by subtracting the exponent of the divisor from that of the dividend, the fraction becomes $a^{1-2} = a^{-1}$. These two values of the fraction must be equal, and hence, $\frac{1}{a} = a^{-1}$.

Again, take the fraction $\frac{a}{a^3}$, in which the exponent of the denominator exceeds that of the numerator by 2. The two values of this fraction, obtained as in the preceding example, are $\frac{1}{a^2}$ and a^{-2}; therefore, $\frac{1}{a^2} = a^{-2}$. In like manner, $\frac{1}{a^3} = a^{-3}$; $\frac{1}{a^4} = a^{-4}$; $\frac{1}{a^5} = a^{-5}$; and, in general, $\frac{1}{a^m} = a^{-m}$; $\frac{1}{(a+b)^m} = (a+b)^{-m}$.

Hence, unity divided by any quantity, is equal to the same quantity with its exponent taken negatively.

Upon the principle just explained, *the denominator of a fraction, or any factor of the denominator, may be transferred to the numerator, care being taken to change the sign of the exponent of the quantity thus transferred.*

Thus, $\frac{ab}{c^2 d^2} = ab \cdot \frac{1}{c^2} \cdot \frac{1}{d^2} = ab \cdot c^{-2} \cdot d^{-2} = abc^{-2}d^{-2}$; also, $\frac{3am^2}{4bc^3} = \frac{3am^2}{4} \cdot \frac{1}{b} \cdot \frac{1}{c^3} = \frac{3am^2}{4} \cdot b^{-1} \cdot c^{-3} = \frac{3am^2b^{-1}c^{-3}}{4}$.

It is evident, on the other hand, *that any factor of the numerator having a negative exponent, may be carried to the denominator, if the sign of that exponent be changed*; *or, when any quantity, integral in form, contains factors having negative exponents, we may convert them into a denominator, observing merely to change the signs of the exponents.*

Thus, $\frac{3a^{-2}b^{-5}}{4} = \frac{3}{4a^2b^5}$; also, $a^{-3}b^{-2}m^{-4} = \frac{1}{a^3b^2m^4}$.

Art. **134.** The fundamental operations are performed on quantities with negative exponents, in the same manner as if the exponents were positive, care being taken with regard to the rules for the signs.

Let it be required to multiply $3a^2d$ by $\frac{c}{d^2}$.

By the usual mode of multiplication, the result would be $\frac{3a^2cd}{d^2} = \frac{3a^2c}{d}$, which is the same as $3a^2cd^{-1}$. But by transferring d^2 to the numerator, and then multiplying, we have $3a^2d \cdot cd^{-2} = 3a^2cd^{1-2} = 3a^2cd^{-1}$, the same as before.

In like manner, $\frac{3a^2x}{4m^2} \cdot \frac{8m^3x^2}{9a^3} = \frac{3a^2m^{-2}x}{4} \cdot \frac{8a^{-3}m^3x^2}{9} = \frac{3 \cdot 8a^{-1}mx^3}{4 \cdot 9} = \frac{2a^{-1}mx^3}{3}$.

Divide $\frac{4bc}{m^3n^2x}$ by $3m^2nx^2$.

By the usual method, we have $\frac{4bc}{3m^5n^3x^3}$. By the use of negative exponents, $\frac{4bc}{m^3n^2x} \div 3m^2nx^2 = 4bcm^{-3}n^{-2}x^{-1} \div 3m^2nx^2 = \frac{4}{3}bcm^{-5}n^{-3}x^{-3}$.

Also, $\frac{3a}{4xy} \div \frac{7x^2y^2}{3a^3} = \frac{3a}{4xy} \cdot \frac{3a^3}{7x^2y^2} = \frac{3ax^{-1}y^{-1}}{4} \times \frac{3a^3x^{-2}y^{-2}}{7} = \frac{9a^4x^{-3}y^{-3}}{28}$.

The third power of $a^{-1}b^2$ is $a^{-3}b^6$; the fourth root of $a^{-1}b^2c^3$ is $a^{-\frac{1}{4}}b^{\frac{1}{2}}c^{\frac{3}{4}}$.

Let the learner perform the following questions, observing that, in the multiplication of fractions, all the factors of the denominator, except such as are numerical, are to be transferred to the numerator, and then the operation may be performed as usual; and that, in dividing by a fraction, the divisor is to be inverted, and then the process is the same as in multiplication. Afterwards any numerical factors, common to numerator and denominator, are to be suppressed.

1. Multiply m^4n^{-3} by $7m^3n^{-2}x^2$.
2. Multiply $3a^{-4}b^{-2}c^3$ by $4a^{-2}b^{-3}c^2$.
3. Multiply $15a^{\frac{1}{3}}b^{\frac{1}{2}}c$ by $2a^{-\frac{1}{6}}b^{-\frac{1}{10}}c^{-\frac{1}{5}}$.
4. Multiply 10^5abc by $10^{-3}a^{\frac{1}{5}}b^2c^{\frac{1}{3}}$.
5. Multiply $3^{-1}a^{-2}b^{-3}$ by $3a^2b^3$.
6. Multiply $\frac{3a^2m^2}{4b^2c^2}$ by $3b^2c^3m$.
7. Multiply $7m^4r^3$ by $\frac{2a^3xy}{m^5r^4}$.
8. Multiply $\frac{7abc^2}{3m^4}$ by $\frac{21mx^2y}{5a^4b^2c^5}$.
9. Multiply $\frac{3(a+b)^2}{c-d}$ by $\frac{4(c-d)^2}{9(a+b)^5}$.
10. Multiply $\frac{35a^3(m+n)^4(c-d)}{4b^2(x+y)}$ by $\frac{28b^3(x+y)^3(m-n)}{25(m+n)^5(c-d)^2}$.
11. Divide $6a^2m^3n$ by $12a^4m^4n$.
12. Divide $5a^5x^{-7}y^3$ by $3a^2x^2y^5$.
13. Divide $\frac{3a^2x^3m^4}{4bc}$ by $3a^{-3}x^{-4}m^3$.
14. Divide $4abc^2x^5$ by $\frac{4mx}{3a^2b^2}$.

15. Divide $\frac{3a^2x^2}{4b^3c^4}$ by $\frac{9a^3x^3}{28bc^2}$.

16. Divide $\frac{7(a+b)^4(c-d)}{3(x+y)}$ by $\frac{2(a+b)^3(c-d)^2}{9(x+y)^2}$.

17. Divide $\frac{56a^2(m+n)^3}{25(2c-3d)^7}$ by $\frac{28a^3(m+n)^4(x-y)}{55(2c-3d)^4}$.

18. Find the 2d power of $3a^{-1}b^{-2}c^3$.

19. Find the 3d power of $4a^{-2}b^2c^{-3}$.

20. Find the 4th power of $2a^{\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{2}{7}}$.

21. Find the 3d power of $10m^{-\frac{1}{6}}x^{-\frac{1}{3}}y^2$.

22. Find the 2d root of $16a^{-2}b^{-4}c^6$.

23. Find the 3d root of $8a^{-9}b^3c^{-6}$.

24. Find the 4th root of $81ab^{-1}c^{-2}d^{-4}$.

25. Find the 3d root of $27a^{\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{1}{4}}$.

26. Find the 5th root of $3a^{-\frac{1}{2}}xy^{-\frac{5}{7}}m^{-\frac{1}{8}}$.

SECTION XLVII.

INEQUALITIES.

Art. **135.** Any expression which indicates that one of two quantities is greater than the other, is called an *inequality.* Thus, $a > b$, which is read a *greater than* b, and $m < n$, which is read m *less than* n, are inequalities.

As inequalities frequently occur in mathematics, it is proper to introduce here some explanation of them.

It is to be remarked, that, although strictly speaking no quantity can be less than zero, yet, in the theory of inequalities, it is convenient to consider negative quantities less than zero, positive quantities being considered greater than zero. Moreover, a negative quantity is said to be so much the less, in proportion as its absolute value is greater. Thus, $0 > -2$, and $-3 > -7$.

With a few exceptions, the principles established relative to equations, are also applicable to inequalities. We shall proceed to notice these principles and the exceptions.

The quantities separated by the sign $>$, are called *members of the inequality*. An inequality is said to continue in the *same sense*, when that member which was the greater previous to a particular operation, continues so afterwards; and two inequalities are said to exist in the *same sense with regard to each other*, when the corresponding members are the greater members. Thus, $a > b$ and $c > d$ exist in the same sense, because the first member of each is greater than its second.

1. *The same quantity or equal quantities may be added to both members, or subtracted from both members of an inequality, and the inequality will continue in the same sense as before.*

Thus, if $5 > 3$, by adding 4, we have $5 + 4 > 3 + 4$, or $9 > 7$; also, if $a > b$, we have $a + c > b + c$. Again, if $-3 > -7$, by adding 8, we have $8 - 3 > 8 - 7$, or $5 > 1$; also, if $-a > -b$, we have $c - a > c - b$.

Moreover, the inequalities $10 > 7$, and $a > b$, give, by subtraction, $10 - 5 > 7 - 5$, or $5 > 2$, and $a - c > b - c$.

Hence, we may transpose from one member to the other any term of an inequality, taking care to change its sign; because that is equivalent to subtracting the same quantity from both members, or adding the same quantity to both members. Thus, if $3x + 20 > 40 - x$, we have, by transposition, $3x + x > 40 - 20$, or $4x > 20$.

2. *The corresponding members of two or more inequalities, existing in the same sense with respect to each other, may be added, and the resulting inequality will exist in the same sense as the given inequalities.*

Thus, by adding the two inequalities $5 > 3$ and $15 > 7$, we have $5 + 15 > 3 + 7$, or $20 > 10$. Also, if $a > b$, $c > d$, and $e > f$, we have $a + c + e > b + d + f$.

3. *But if two inequalities existing in the same sense, be subtracted, member from member, the resulting inequality will not always exist in the same sense as the given inequalities.*

Indeed the result may, according to circumstances, be an inequality in the same sense as those given, or one in a different sense, or it may be an equation.

Thus, $13 > 4$ and $20 > 7$ give, by subtraction, $20 - 13 > 7 - 4$, or $7 > 3$, which is an inequality in the same sense as the two proposed.

Again, $15 > 12$ and $10 > 3$ give, by subtraction, $15 - 10 < 12 - 3$, or $5 < 9$, an inequality in the opposite sense to the proposed.

Finally, $20 > 17$ and $12 > 9$ give $20 - 12 = 17 - 9$, or $8 = 8$, an equation.

In general, let $a > b$ and $c > d$; then, according to the particular values of a, b, c and d, we may have $a - c > b - d$, $a - c < b - d$, or $a - c = b - d$.

4. *The two members of an inequality may be multiplied or divided by the same positive quantity, or by equal positive quantities, and the result will be an inequality in the same sense as the proposed.*

For example, multiplying both members of $11 > 7$ by 8, we have $88 > 56$. Also, if $a > b$, $ac > bc$.

Again, dividing both members of $35 > 21$ by 7, we have $5 > 3$. Also, if $am > cm$, we have $a > c$; and if $m > n$, $\frac{m}{p} > \frac{n}{p}$.

5. *But, if both members of an inequality be multiplied by the same negative quantity, or by equal negative quantities, the result will be an inequality in a sense opposite to that of the proposed.*

Thus, if $7 > 5$, and we multiply by -3, we have $-21 < -15$. Also, if $a > b$, multiplying by $-m$, we have $-am < -bm$. In these examples, the sense is inverted, because a negative quantity is less in proportion as its absolute value is greater.

6. *Hence it follows, that the sense of an inequality will be in-*

verted, if all the signs of both members be changed; because this is the same as multiplying both members by -1.

7. *Both members of an inequality, if they are positive quantities, can be raised to the same power, and the result will be an inequality in the same sense as the proposed.*

Thus, from $7 > 2$ we have $7^2 > 2^2$, or $49 > 4$; and from $a > b$, $a^m > b^m$.

8. *But if both members of an inequality are not positive, and both be raised to the same power denoted by a whole number, the resulting inequality will not always exist in the same sense as the proposed.*

Thus, $3 > -2$ gives $3^2 > (-2)^2$, or $9 > 4$, in the same sense as the proposed. But $-3 > -5$ gives $(-3)^2 < (-5)^2$, or $9 < 25$, in the reverse sense of the proposed.

9. *Roots to the same degree, of the two members of an inequality, may be extracted, and the resulting inequality will be in the same sense as the proposed.*

Thus, $27 > 8$ gives $\sqrt[3]{27} > \sqrt[3]{8}$, or, $3 > 2$, and, in general, $a > b$ gives $\sqrt[n]{a} > \sqrt[n]{b}$.

If the root be of an even degree, it is necessary that both members of the given inequality be positive; otherwise one or both of the roots would be imaginary, and they could not be compared.

Art. **136**. There are some problems, the solution of which involves the principles of inequalities. The following are of this kind.

1. Three times a certain number added to 16, exceeds twice that number added to 24, and two fifths of the number added to 5 is less than 11. Required the number.

Let x represent the number; then $3x + 16 > 2x + 24$, and $\frac{2x}{5} + 5 < 11$. The first inequality, by transposition and reduction, gives $x > 8$. The second, multiplied by 5, becomes

$2x + 25 < 55$, which, by transposition, reduction, and division, gives $x < 15$. Any number, therefore, entire or fractional, which is greater than 8 and less than 15, will fulfil the conditions of the question.

2. Says A to B, I have an exact number of dollars in my purse; if I had twice as many and \$10, I should have more than \$49; but if I had three times as many, my number would be less than the number I now have increased by \$41. Required the number of dollars in his purse.

3. A certain number divided by 17 gives an entire quotient, which quotient increased by 2, exceeds 4; but if the number be multiplied by 2, and the product be increased by 4, the result will be less than the number itself increased by 56. What is the number?

SECTION XLVIII.

EQUIDIFFERENCE.

Art. **137**. The difference between two quantities is sometimes called their *arithmetical ratio*, or *ratio by subtraction*. Thus, the arithmetical ratio of 9 to 7 is $9 - 7$ or 2, and that of a to b is $a - b$.

Four quantities, such that the difference between the first and second, is the same as that between the third and fourth, constitute what is called *an equidifference*, sometimes called also *an arithmetical proportion*. Thus, 9, 7, 5 and 3 form an equidifference; for $9 - 7 = 5 - 3$. This is sometimes expressed thus, $9 . 7 : 5 . 3$, in which one point denotes difference, and two points denote equality. But this notation is objectionable, because these characters are sometimes used, the one to represent multiplication, and the other division.

In like manner, if the quantities, a, b, c and d, are such that $a - b = c - d$, or $-a + b = -c + d$, these four quantities constitute an equidifference.

The quantities, a, b, c, d, are called *terms* of the equidifference. Also, a and d, the first and last terms, are called the *extremes*, because they occupy the extremities; b and c, the second and third terms, are called the *means*, because they occupy the middle in the equidifference.

Remark. In the definition of equidifference, it is supposed, that, if the second term is greater than the first, the fourth is greater than the third; but, if the second is less than the first, the fourth is less than the third.

From any equidifference, $a - b = c - d$, or, $-a + b = -c + d$, we deduce, by transposition, $a + d = b + c$; also, $a = b + c - d$, and $d = b + c - a$, $b = a + d - c$, and $c = a + d - b$. Hence,

In any equidifference, the sum of the means is equal to the sum of the extremes. Moreover, either mean is equal to the sum of the extremes, diminished by the other mean; and either extreme is equal to the sum of the means, diminished by the other extreme.

Suppose we have $a + d = b + c$; by transposition we obtain $a - b = c - d$.

Therefore, if the sum of two quantities is equal to the sum of two other quantities, the first two may be made the means, and the last two the extremes, or the reverse, of an equidifference.

When three quantities, a, b, c, either increasing or decreasing, are such that the difference between the first and second is equal to that between the second and third, that is, $a - b = b - c$, they constitute what is called a *continued equidifference*, and the quantity b is called the *arithmetical mean* between a and c. Thus, 3, 5, and 7, or 12, 8, and 4 form a continued equidifference.

Take, for example, $a - b = b - c$. From this we deduce $b = \frac{a + c}{2}$; also, $a = 2b - c$, and $c = 2b - a$. Hence,

In any continued equidifference, the mean is half the sum of the extremes, and either extreme is found by subtracting the other extreme from twice the mean.

1. The means of an equidifference are 10 and 12, and the known extreme is 6. Required the other extreme.

2. The extremes of an equidifference are 7 and $4\frac{1}{2}$, and one of the means is 6. What is the other mean?

3. The means of an equidifference are 8 and 12, and the last term exceeds twice the first by 5. Required the extremes.

4. In a continued equidifference, the extremes are 10 and $15\frac{1}{2}$. Required the mean.

5. In a continued equidifference, the mean is 7 and one extreme is 8. Required the other extreme.

6. The mean of a continued equidifference is 14, and the third term exceeds the first by 8. Required the extremes.

SECTION XLIX.

RATIO AND PROPORTION.

Art. **138**. The quotient arising from the division of one quantity by another, whether the division can be exactly performed or can only be expressed, is called the *ratio* of these quantities. It is sometimes called *ratio by division*, or *geometrical ratio.* But when the word *ratio* simply is used, it signifies ratio resulting from division.

A ratio is most appropriately expressed in the form of a fraction. Thus, $\frac{3}{5}$ is the ratio of 3 to 5, and $\frac{a}{b}$ is that of a to b.

An equation formed by two equal ratios, is called a *proportion.* Sometimes the term *geometrical proportion* is used, to distinguish it from *arithmetical proportion* or equidifference. Thus, $\frac{3}{7}=\frac{9}{21}$, and $\frac{a}{b}=\frac{c}{d}$ are proportions.

For the sake of convenience in writing and printing, most authors express division by the sign : , placed between the quantities, and, instead of the sign $=$, use the sign : : . Thus, $a:b::$

$c:d$ is read, a is to b as c is to d, and is the same as $\frac{a}{b} = \frac{c}{d}$. The signification in both cases, is, that a divided by b, is equal to c divided by d. In this treatise points will sometimes be used to denote division, but the sign $=$ will always be preferred rather than $::$.

In any proportion, $a:b = c:d$, the quantities a, b, c, and d, are called the *terms* of the proportion. The two quantities a and b are called the *terms* of the first ratio, c and d those of the second.

Moreover, a and c are called the *antecedents* of the proportion, a being the antecedent of the first ratio, and c that of the second; b and d are called the *consequents* of the proportion, b being the consequent of the first ratio, and d that of the second. Also, a and d are called the *extremes*, b and c the *means* of the proportion.

These names are derived from the position in which the terms stand with respect to each other, when the division is indicated by points. *Antecedent* signifies going before, and *consequent*, following after. Thus, in the ratio $a:b$, a precedes and b follows after it. The signification of the words *means* and *extremes* has already been explained.

Art. **139**. There are several important properties of proportions, which we shall now proceed to demonstrate.

1. Take any proportion, $a:b = c:d$, or $\frac{a}{b} = \frac{c}{d}$. If we multiply the proportion in its second form, by the denominators b and d, we have $ad = bc$. But a and d are the extremes, and b and c the means. Hence,

In any proportion the product of the means is equal to the product of the extremes.

2. Suppose we have $ad = bc$. Dividing both members by b and d, we have $\frac{a}{b} = \frac{c}{d}$, or $a:b = c:d$. Hence,

If the product of two quantities is equal to the product of two other quantities, the two factors of either product may be made the means, and the two factors of the other product, the extremes of a proportion.

3. Any three terms of a proportion being given, we can always find the remaining one. For, take any proportion, $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$, which gives $a\,d = b\,c$; hence, by division, $a = \frac{b\,c}{d}$, $d = \frac{b\,c}{a}$, $b = \frac{a\,d}{c}$, and $c = \frac{a\,d}{b}$. Therefore,

In any proportion, either mean is equal to the product of the extremes, divided by the other mean; and either extreme is equal to the product of the means, divided by the other extreme.

From this it follows, that,

If three terms of one proportion are respectively equal to the three corresponding terms of another proportion, the remaining term of one must be equal to the remaining term of the other.

4. The proportion, $a : b = b : c$, in which the two mean terms are the same, is called a *continued proportion*, and b is called a *mean proportional* between a and c. This proportion gives $b^2 = a\,c$, and $b = \sqrt{a\,c}$. Hence,

The mean proportional between two quantities, is equal to the second root of their product.

From this it follows, that,

If the second power of any quantity is equal to the product of two others, the first quantity is a mean proportional between the last two.

For the equation, $a\,c = b^2$, gives $a : b = b : c$.

5. Let there be given $a : b = c : d$. (1)

This produces $a\,d = b\,c$. (A)

Dividing both members of equation (A) by c and d, we have

$$\frac{a}{c} = \frac{b}{d}, \text{ or } a : c = b : d. \quad (2)$$

Dividing equation (A) by a and b,

$$\frac{d}{b} = \frac{c}{a}, \text{ or } d : b = c : a. \quad (3)$$

Changing the order of the ratios in proportion (1),

$$c : d = a : b. \quad (4)$$

Changing equation (A) member for member, and then dividing by a and c,

$$\frac{b}{a} = \frac{d}{c}, \text{ or } b : a = d : c. \quad (5)$$

Comparing proportions (2), (3), (4), and (5), with the given proportion (1), we infer, that,

In any proportion the means may exchange places; the extremes may exchange places; the extremes may be made the means, and the means the extremes; both ratios may, at the same time, be inverted, that is, the antecedent and consequent of each ratio may exchange places.

Indeed, in a given proportion, any change may be made in the order of the terms, provided that, in each arrangement, the product of the means being put equal to the product of the extremes, the same equation is produced, as that arising from the given proportion. The same proportion, therefore, admits of eight forms, viz:

$$a : b = c : d; \; a : c = b : d;$$
$$d : b = c : a; \; d : c = b : a;$$
$$b : a = d : c; \; b : d = a : c;$$
$$c : a = d : b; \; c : d = a : b;$$

for each proportion gives $a d = b c$.

6. Since the value of a fraction is not changed, when both numerator and denominator are either multiplied or divided by the same quantity, it follows, that,

In a proportion, we may multiply or divide both terms of either ratio by the same quantity, and we may multiply or divide all the terms of a proportion by the same quantity, without disturbing the proportion. We may also multiply or divide both terms of the

first ratio by one quantity, and both terms of the second ratio by another quantity, or we may multiply both terms of one ratio by any quantity, and divide both terms of the other ratio by the same or a different quantity, without disturbing the proportion.

Thus, if $a:b=c:d$, or $\frac{a}{b}=\frac{c}{d}$, we have

$$am:bm=c:d;\ a:b=cn:dn;$$
$$\frac{a}{m}:\frac{b}{m}=c:d;\ a:b=\frac{c}{n}:\frac{d}{n};$$
$$am:bm=cm:dm;\ \frac{a}{m}:\frac{b}{m}=\frac{c}{m}:\frac{d}{m}.$$

Also, $am:bm=cn:dn$; $\frac{a}{m}:\frac{b}{m}=\frac{c}{n}:\frac{d}{n}$;

$$am:bm=\frac{c}{n}:\frac{d}{n};\ \frac{a}{m}:\frac{b}{m}=cn:dn.$$

7. *Both of the antecedents or both of the consequents of a proportion, may either be multiplied or divided by the same quantity or by equal quantities, without disturbing the proportion.*

Thus, if $a:b=c:d$, or $\frac{a}{b}=\frac{c}{d}$, we have

$$\frac{am}{b}=\frac{cm}{d},\text{ or } am:b=cm:d;$$
$$\frac{a}{bn}=\frac{c}{dn},\text{ or } a:bn=c:dn.$$

Also, $\frac{a}{m}:b=\frac{c}{m}:d$; $a:\frac{b}{n}=c:\frac{d}{n}$.

The reason is obvious, for these several results are produced by multiplying or dividing equal fractions by the same quantity.

8. Suppose we have the two proportions,

$$a:b=c:d,\text{ and } a:b=m:n.$$

Then, according to ax. 7, we have

$c:d=m:n$. Hence,

If two proportions have a common ratio, or a ratio in one proportion equal to a ratio in the other, the two remaining ratios are equal, and may form a proportion.

9. Suppose we have the two proportions

$a : b = c : d$, and $a : m = c : n$, in which the antecedents are alike. By changing the means in each, we have

$a : c = b : d$, and $a : c = m : n$; consequently, on account of the common ratio, $a : c$, we have

$b : d = m : n$; hence, $b : m = d : n$. Therefore,

If in two proportions, the antecedents are alike or equal, the consequents will form a proportion.

Suppose now that we have

$a : b = c : d$, and $m : b = n : d$, two proportions in which the consequents are alike. Changing the means in each, we have

$a : c = b : d$, and $m : n = b : d$. Consequently, on account of the common ratio,

$a : c = m : n$; hence, $a : m = c : n$. Therefore,

If in two proportions the consequents are alike or equal, the antecedents will form a proportion.

10. Suppose $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$.

Adding to or subtracting from both members of the equation any quantity m, and reducing to a common denominator, we have

$$\frac{a \pm bm}{b} = \frac{c \pm dm}{d}, \text{ or } a \pm bm : b = c \pm dm : d.$$

The last proportion becomes, by changing the means,

$$a \pm bm : c \pm dm = b : d = a : c \quad (1);$$

since, from the given proportion, these last two ratios are equal.

If, in the given proportion, $a : b = c : d$, both ratios be inverted, the proportion becomes

$$b : a = d : c, \text{ or } \frac{b}{a} = \frac{d}{c}.$$

Adding to or subtracting from both members of this equation any quantity m, and reducing to a common denominator, we have

$\frac{b \pm am}{a} = \frac{d \pm cm}{c}$, or $b \pm am : a = d \pm cm : c$, which, if the means be changed, becomes

$b \pm am : d \pm cm = a : c = b : d.$ (2)

Comparing proportions (1) and (2) with the given proportion, we infer, that,

In any proportion, the first antecedent plus or minus any number of times its consequent, is to the second antecedent plus or minus the same number of times its consequent, also the first consequent plus or minus any number of times its antecedent, is to the second consequent plus or minus the same number of times its antecedent, as the first term is to the third, or as the second is to the fourth.

11. From the proportion $b \pm am : d \pm cm = a : c$, which was obtained above, we have, by taking the plus sign,

$b + am : d + cm = a : c$; by taking the minus sign,
$b - am : d - cm = a : c$; hence,
$b + am : d + cm = b - am : d - cm$; making $m = 1$,
$b + a : d + c = b - a : d - c$; changing the means,
$b + a : b - a = d + c : d - c.$

From the last two proportions, we infer, that,

In any proportion, the sum of the first two terms is to the sum of the last two, as the difference of the first two terms is to the difference of the last two; also, the sum of the first two terms is to their difference, as the sum of the last two terms is to their difference.

Remark. It is manifest that the last two proportions might be written thus:

$a + b : c + d = a - b : c - d$, and
$a + b : a - b = c + d : c - d$; for these are deduced from those proportions, in one case, by changing the signs of both numerator and denominator of a fraction, and, in the other, by changing the signs of both members of an equation.

12. Let the proportion, $a : b = c : d$, be given.

By changing the means, we have

$a : c = b : d$, or $\frac{a}{c} = \frac{b}{d}$; whence,

$\frac{a \pm cm}{c} = \frac{b \pm dm}{d}$, or $a \pm cm : c = b \pm dm : d$.

Changing the means in the last proportion,

$a \pm cm : b \pm dm = c : d = a : b$. (1)

By making the means the extremes, and the extremes the means, in the given proportion, we have

$c : a = d : b$, or $\frac{c}{a} = \frac{d}{b}$; whence,

$c \pm am : a = d \pm bm : b$; changing the means,

$c \pm am : d \pm bm = a : b = c : d$. (2)

Comparing proportions (1) and (2) with the given proportion, we infer, that,

In any proportion, the first antecedent plus or minus any number of times the second, is to the first consequent plus or minus the same number of times the second, also the second antecedent plus or minus any number of times the first, is to the second consequent plus or minus the same number of times the first, as either antecedent is to its consequent.

13. By making $m = 1$, in proportion (2) of number 12, we have

$c \pm a : d \pm b = a : b = c : d$ (1); taking the sign $+$,

$c + a : d + b = a : b$; taking the sign $-$,

$c - a : d - b = a : b$; whence,

$c + a : d + b = c - a : d - b$ (2); changing the means,

$c + a : c - a = d + b : d - b$ (3).

The last two proportions may evidently become

$a + c : b + d = a - c : b - d$, and

$a + c : a - c = b + d : b - d$.

Comparing proportions (1), (2), and (3) with the given proportion, $a : b = c : d$. we infer, that,

In any proportion, the sum or difference of the antecedents, is to the sum or difference of the consequents, as either antecedent is to its consequent; the sum of the antecedents is to the sum of the consequents, as the difference of the antecedents is to the difference of the consequents; also, the sum of the antecedents is to their difference, as the sum of the consequents is to their difference.

14. *If in any proportion, the antecedents are alike or equal, the consequents must be equal; also, if the consequents are alike or equal, the antecedents must be equal.*

For equal fractions having equal numerators, must have equal denominators; and equal fractions having equal denominators, must have equal numerators.

Thus, if $a:b=a:m$, then $b=m$; or if $a:b=c:m$, and $a=c$, then $b=m$. Also, $a:m=c:m$ gives $a=c$, and $a:b=c:m$ gives, upon the supposition that $b=m$, $a=c$.

It is moreover evident, that,

If the second term is greater than the first, the fourth must be greater than the third, and conversely; and if the first two terms are equal, the last two must also be equal.

15. Suppose we have a series of equal ratios, viz:

$$a:b=c:d=e:f=g:h, \text{ or } \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$$

Let q represent the value of each of these fractions; then,

$\frac{a}{b}=q, \frac{c}{d}=q, \frac{e}{f}=q, \frac{g}{h}=q.$ By multiplication,
$a=bq$, $c=dq$, $e=fq$, $g=hq$. Adding these equations
$a+c+e+g=bq+dq+fq+hq$, or
$a+c+e+g=(b+d+f+h)q$ Dividing by $b+d+f+h$,
$\frac{a+c+e+g}{b+d+f+h}=q=\frac{a}{b}=\frac{c}{d}$, &c.; or
$a+c+e+g:b+d+f+h=a:b=c:d=e:f=g:h.$

Hence,

In any series of equal ratios, the sum of the antecedents is to the sum of the consequents, as any one of the antecedents is to its consequent.

16. If $a : b = c : d$, and $e : f = g : h$, that is, $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$, by multiplying these two equations together, we obtain $\frac{ae}{bf} = \frac{cg}{dh}$, or $ae : bf = cg : dh$.

The same result would have been obtained by multiplying together the corresponding terms of the two given proportions. This is called multiplying the proportions *in order*. Hence,

If two or more proportions are multiplied in order, the result will form a proportion.

From this it follows, that,

If proportions are divided in order, the result will form a proportion.

17. Given, $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$.

Raising both members to any power denoted by m, we have

$$\frac{a^m}{b^m} = \frac{c^m}{d^m}, \text{ or } a^m : b^m = c^m : d^m.$$

Therefore,

Similar powers of proportional quantities form a proportion.

From this it follows, that,

Similar roots of proportional quantities will form a proportion.

SECTION L.

PROGRESSION BY DIFFERENCE.

Art. **140.** A series of quantities, such that each is greater than that which immediately precedes it, or such that each exceeds that which immediately follows it, by the same quantity, constitutes what is called a *progression by difference*, or *arithmetical progression.*

Thus, the natural numbers, 1, 2, 3, 4, &c., form such a progression, since the difference between any two contiguous numbers is unity.

Progression by difference may be either *increasing* or *decreasing*. The series 3, 5, 7, 9, &c. is an *increasing*, but 20, 18, 16, 14, &c. is a *decreasing progression.*

To exhibit a progression generally, let a be the first term, and d the common difference; then, if the progression be increasing, $\overset{\text{1st}}{a}$, $\overset{\text{2d}}{(a+d)}$, $\overset{\text{3d}}{(a+2d)}$, $\overset{\text{4th}}{(a+3d)}$, $\overset{\text{5th}}{(a+4d)}$, &c., will be successive terms at the commencement of the series; or, if the progression be decreasing, $\overset{\text{1st}}{a}$, $\overset{\text{2d}}{(a-d)}$, $\overset{\text{3d}}{(a-2d)}$, $\overset{\text{4th}}{(a-3d)}$, $\overset{\text{5th}}{(a-4d)}$, &c., will be the successive terms.

By examining these series, we perceive, that if the progression is increasing, the second term is found by adding once the common difference to the first term; the third term is found by adding twice the common difference to the first term; the fourth, by adding three times, and the fifth, by adding four times the common difference, to the first term. But, if the progression is decreasing, we subtract from the first term, once, twice, three times, four times the common difference, to find the second, third, fourth, fifth terms. That is, in all cases, the difference is multiplied by a number less by 1 than that which marks the place of the term, and the product is either added to, or subtracted from, the first term. Hence, if in addition to the notation used above,

n denote the number of terms, and l the last term, we shall have the formula

$l = a + (n-1)d$, in an increasing progression; and
$l = a - (n-1)d$, in a decreasing progression.

If we use the double sign $\pm$, the general formula for the last term is,

$l = a \pm (n-1)d$. Therefore,

To find the last term, multiply the common difference by the number of terms minus one, and add the product to the first term if the progression is increasing, or subtract the product from the first term if the progression is decreasing.

Ex. 1. Find the 10th term of the progression, 3, 7, 11, &c.

In this example, $a=3$, $d=4$, and $n=10$; by substituting these values in the formula, $l=a+(n-1)d$, we have $l=3+(10-1)4=3+9.4=3+36=39$. The last term therefore is 39.

Ex. 2. Find the 8th term of the series, 50, 48, 46, &c.

We have, in this case, $l=50-(8-1)2=50-7.2=50-14=36$.

Art. **141**. We wish now to find a formula for the sum of any number of terms in progression by difference. For this purpose, let S denote the sum of all the terms in the progression, a, $a+d$, $a+2d$, &c. Then,

$$S = \overset{\text{1st}}{a} + \overset{\text{2d}}{(a+d)} + \overset{\text{3d}}{(a+2d)} + \overset{\text{4th}}{(a+3d)} + \ldots\ldots\ldots + \overset{n\text{th}}{l}. \quad (1)$$

If we begin with the last term, it is evident that the successive terms of the same progression will be l, $l-d$, $l-2d$, $l-3d$, &c. Hence,

$$S = \overset{n\text{th}}{l} + \overset{(n-1)\text{th}}{(l-d)} + \overset{(n-2)\text{th}}{(l-2d)} + \overset{(n-3)\text{th}}{(l-3d)} + \ldots\ldots\ldots + \overset{\text{1st}}{a}. \quad (2)$$

Remark. The terms cannot all be written, unless some definite value be given to n. Points are therefore used, to supply the place of the indefinite number of terms which are omitted.

Adding the equations (1) and (2), we have

$$2S=(a+l)+(a+l)+(a+l)+(a+l)+\ldots+(a+l);$$

or, since the quantities included in the several parentheses are the same, and since n represents the number of terms,

$$2S=n(a+l); \text{ and, therefore,}$$

$$S=\frac{n(a+l)}{2}. \text{ Hence,}$$

The sum of any number of terms in progression by difference, is found, by multiplying the sum of the first and last terms by half the number of terms, or by multiplying half the sum of the first and last terms by the number of terms.

By substituting the value of l in the formula just found, we can obtain another formula for S. For, since $l=a\pm(n-1)d$, we have $S=\frac{n[a+a\pm(n-1)d]}{2}=\frac{2na\pm n(n-1)d}{2}$, or, $S=na\pm\frac{nd(n-1)}{2}$.

In the first formula for the sum, it is necessary to find l before we can find S; but, by the second formula, S can be found, when a, d and n are known, without previously finding l.

Ex. Find the sum of 12 terms of the series, 7, 9, 11, &c.

In this example, $a=7$, $d=2$, and $n=12$. We first find the last term and then the sum. By substituting in the formula for l, we have $l=7+(12-1)2=29$. Then substituting in the first formula for S, we have $S=\frac{12(7+29)}{2}=6\,.\,36=216$.

By using the second formula for S, we have $S=12\,.\,7+\frac{12\,.\,2\,.\,11}{2}=12\,.\,7+12\,.\,11=12(7+11)=12\,.\,18=216$.

N. B. The two formulæ, $l=a\pm(n-1)d$, and $S=\frac{n(a+l)}{2}$, should be retained in memory by the learner.

Art. **142**. The first and last terms of a progression are called the *extremes;* and, when the number of terms is odd, the middle

one is called the *mean*, but when the number of terms is even, the two situated midway between the extremes, are called the *means*.

If we observe the process of adding equations (1) and (2), in Art. **141**, it will be manifest, *that the sum of any two terms equally distant from the extremes, is the same as the sum of the extremes.*

Moreover, if the number of terms is odd, the sum of the extremes will be equal to twice the mean. For, the number of terms being odd, the middle terms of equations (1) and (2) will be of the same value, although expressed in one by a plus a certain number of times the difference, and in the other by l minus the same number of times the difference. These two middle terms therefore being added, their sum will be the same as twice one of them.

Art. **143**. The two equations, $l = a + (n-1)d$, and $S = \frac{n(a+l)}{2}$, involve five different quantities, any three of which being given, the remaining two can be found.

There may arise then the ten following problems, viz:

1. Given a, d and n, find l and S.
2. Given a, d and l, find n and S.
3. Given a, n and S, find d and l.
4. Given a, l and S, find d and n.
5. Given a, n and l, find d and S.
6. Given a, d and S, find n and l.
7. Given d, n and S, find a and l.
8. Given d, n and l, find a and S.
9. Given n, l and S, find a and d.
10. Given d, l and S, find a and n.

The first of these problems has already been solved, and the equations which we have found for l and S, may be assumed, in the solution of the other nine problems. Of these last we shall solve the fifth, and leave the others to be performed by the learner.

The problem is, to find d and S, when a, l and n are known.

The value of S is already given, viz: $S = \frac{n(a+l)}{2}$; and the

equation $l = a + (n - 1)d$, gives, by transposition and division, $d = \frac{l - a}{n - 1}$.

Art. **144**. This value of d will enable us to insert any number of terms between two given quantities, a and l, so that the whole series shall form a progression by difference. The quantities thus inserted, are called *mean differentials*, or *arithmetical means*.

Thus, if it be required to insert m mean differentials between the quantities a and l, as there would be $m+2$ terms in the whole, to find the common difference, we have only to substitute $m+2$ instead of n, in the formula for d, which gives $d = \frac{l - a}{m + 2 - 1} = \frac{l - a}{m + 1}$.

Hence,

When a certain number of mean differentials is to be inserted between two quantities, to find the common difference, divide the difference between the quantities by a number greater by one than the number of terms to be inserted.

Knowing the common difference, it is easy to write the progression, which, expressed in general terms, will be as follows, viz:

$$a,\ a + \frac{l - a}{m + 1},\ a + 2\left(\frac{l - a}{m + 1}\right),\ a + 3\left(\frac{l - a}{m + 1}\right),\ \ldots\ldots,\ l;\ \text{or}$$

$$a,\ \frac{ma + l}{m + 1},\ \frac{(m - 1)a + 2l}{m + 1},\ \frac{(m - 2)a + 3l}{m + 1},\ \ldots\ldots\ldots,\ l.$$

As an example, let it be required to insert six mean differentials between 4 and 25.

Here $d = \frac{l - a}{m + 1}$, becomes $d = \frac{25 - 4}{7} = 3$; and the progression is 4, 7, 10, 13, 16, 19, 22, 25.

It is manifest from what precedes, that,

If between the terms of a progression by difference taken two and two, the same number of mean differentials be inserted, the result will be in progression.

For example, let it be required to insert between every two adjacent terms of the progression, 3, 9, 15, 21, two mean differentials.

In this case, $d = \frac{l-a}{m+1}$ becomes $d = \frac{9-3}{3} = 2$; and the progression is 3, 5, 7, 9, 11, 13, 15, 17, 19, 21.

SECTION LI.

EXAMPLES INVOLVING PROGRESSION BY DIFFERENCE.

Art. **145.** 1. How many strokes does a clock strike in 12 hours?

2. Find the 10th term and the sum of the first 10 terms of the series, 20, 25, 30, &c.

3. Find the 16th term and the sum of the first 16 terms in the series, 100, 98, 96, &c.

4. Find the last term and the sum of the series, 12, $13\frac{1}{4}$, $14\frac{1}{2}$, &c. the number of terms being 30.

5. The number of terms being 28, find the last term and the sum of the series, 3, $3\frac{7}{8}$, $4\frac{3}{4}$, &c.

6. Insert six mean differentials between 20 and 55.

7. Insert five mean differentials between 6 and 10.

8. Insert five mean differentials between every two adjacent terms of the progression, 5, 17, 29, 41.

9. Suppose that, as in Venice, a clock denoted, by the number of strokes, the hours from 1 to 24, how many strokes would such a clock strike in 24 hours?

10. A farmer wished to set out, upon a triangular piece of land, 25 rows of apple trees, the first row containing 2 trees, the second 5, the third 8, and so on. How many trees did he require for his purpose?

11. A gardener has 100 plants and a reservoir of water all in a straight line, the plants being 3 feet asunder, and the reservoir 10 feet from the first plant. How far must he walk in order to water these plants, if he commence at the reservoir, and return to it for a new supply of water for each plant, finally coming to the reservoir after having watered the last one?

12. A falling body descends, *in vacuo*, $16\frac{1}{12}$ feet the first second, and in each succeeding second, $32\frac{1}{6}$ feet more than in the preceding. How far will a body descend in 10 seconds?

13. We observe that, in the preceding question, the difference is just double the first term. Let the learner generalize that question, by substituting, in the second formula for S, $2a$ instead of d.

14. Two travelers, A and B, 188 miles asunder, set out at the same time with the intention of meeting. A goes regularly 10 miles per day; but B goes 3 miles the first day, 6 the second, 9 the third, and so on. In how many days will they meet?

15. Two men, 135 miles asunder, set out at different times and travel towards each other. One starts 5 days before the other, and goes 1 mile the 1st day, 2 miles the 2d, 3 miles the 3d, and so on. The other travels 20 miles the 1st day, 18 the 2d, 16 the 3d, and so on. How many days and what distance will each have traveled when they meet?

16. Divide 51 into three parts, which shall form a progression by difference, the common difference being 5.

17. Find three numbers in arithmetical progression, such that their sum shall be 18 and their continued product 192.

Remark. Let $y =$ the common difference, and $x =$ the mean term. Then $x - y$, x, and $x + y$ will represent the numbers.

18. Divide 50 into five parts, which shall form a progression by difference, of which the first term shall be to the last as 7 to 3.

19. There is a number consisting of three digits, which form a decreasing arithmetical progression. The sum of the digits is 9, and if 396 be subtracted from the number, the digits will be inverted. Required the number.

SECTION LII.

PROGRESSION BY QUOTIENT.

Art. **146**. *A progression by quotient*, called also *geometrical progression*, is a series of quantities such, that, if any one of them be divided by the next preceding, the quotient will be the same, in whatever part of the series the two successive terms are taken.

Progression by quotient may be either *increasing* or *decreasing*. Thus, 2, 4, 8, 16, 32 form an increasing, and 60, 20, $\frac{20}{3}$, $\frac{20}{9}$, a decreasing progression by quotient.

The quotient arising from the division of any term by that which precedes it, is called the *common ratio*. The *ratio* in the first of the two progressions given above, is 2, and that in the second is $\frac{1}{3}$.

In general, let a, b, c, d, &c. be the successive terms of a progression.

Let q represent the constant ratio; then, since each term is q times the preceding, we have

$$b = a\,q,\ c = a\,q^2,\ d = a\,q^3,\ e = a\,q^4,\ \text{\&c.}$$

Now representing the last term by l, and supplying by points the place of the indefinite number of terms omitted, the terms of the progression will be,

$$\overset{\text{1st}}{a},\ \overset{\text{2d}}{a\,q},\ \overset{\text{3d}}{a\,q^2},\ \overset{\text{4th}}{a\,q^3},\ \overset{\text{5th}}{a\,q^4},\ \overset{\text{6th}}{a\,q^5},\ \overset{\text{7th}}{a\,q^6},\ \ldots\ldots\ldots\ldots,\ l.$$

We readily perceive, that the exponent of q in any term is ess by unity, than the number which marks the place of that term. Thus, the 4th term is $a\,q^3$, the 5th $a\,q^4$. Consequently, if n represent the number of terms, the nth or last term will be $a\,q^{n-1}$. Therefore, the formula for the last term is

$$l = a\,q^{n-1}.$$

Hence.

Any term of a progression by quotient, may be found, by multiplying the first term by that power of the ratio, denoted by a number 1 *less than that which marks the place of the term.*

Ex. What is the sixth term of the series, 3, 6, 12, &c.?

Here $a = 3$, $q = 2$, and $n = 6$; therefore, $l = a q^{n-1}$ becomes $l = 3 \,.\, 2^5 = 3 \,.\, 32 = 96$.

Ex. 2. Required the 7th term of the series 3645, 1215, 405, &c.

In this question, $a = 3645$, $q = \frac{1}{3}$, $n = 7$. Then $l = 3645 \times (\frac{1}{3})^6 = 3645 \,.\, \frac{1}{729} = 5$.

Art. **147**. To find the sum of any number of terms, denote this sum by S; then

$$S = a + aq + aq^2 + aq^3 + aq^4 + aq^5 + \ldots\ldots + aq^{n-2} + aq^{n-1}.$$

Multiplying this equation by q, we have

$$qS = aq + aq^2 + aq^3 + aq^4 + aq^5 + aq^6 + \ldots\ldots + aq^{n-1} + aq^n.$$

Subtracting the former of these equations from the latter, observing that the terms of the second members all cancel except a and $a q^n$, we obtain

$qS - S = aq^n - a$; or $(q-1)\,S = a\,q^n - a$; consequently,

$$S = \frac{aq^n - a}{q-1} = \frac{a(q^n - 1)}{q-1}.$$

But $\dfrac{aq^n - a}{q-1} = \dfrac{aq^{n-1}q - a}{q-1}$; substituting l instead of its equal, aq^{n-1}, we have $S = \dfrac{lq - a}{q-1}$.

We have then two formulæ for the sum of a geometrical progression, viz:

$$S = \frac{a(q^n - 1)}{q-1}, \text{ and } S = \frac{ql - a}{q-1}. \text{ Hence,}$$

To find the sum of a progression by quotient, subtract unity from that power of the ratio, denoted by the number of terms, multiply the remainder by the first term, and divide the product by the ratio minus unity; or, multiply the last term by the ratio, subtract the first term from the product, and divide the remainder by the ratio minus unity.

Ex. Required the sum of the series, 1, 2, 4, 8, &c., the number of terms being 10?

In this question, $a = 1$, $q = 2$, and $n = 10$. Therefore, $S = \frac{1(2^{10} - 1)}{2 - 1} = \frac{1(1024 - 1)}{2 - 1} = 1023.$

Or we may first find the last term, and then use the formula, $S = \frac{lq - a}{q - 1}$. We have then $l = 1 \cdot 2^9 = 512$, and $S = \frac{2 \cdot 512 - 1}{2 - 1} = \frac{1024 - 1}{1} = 1023.$

Ex. 2. Required the sum of the first six terms of the series, 10, 5, $\frac{5}{2}$, &c.?

Here $a = 10$, $q = \frac{1}{2}$, and $n = 6$. Using the first formula for S, we have $S = \frac{10[(\frac{1}{2})^6 - 1]}{\frac{1}{2} - 1} = \frac{10(\frac{1}{64} - 1)}{\frac{1}{2} - 1} = \frac{10(-\frac{63}{64})}{-\frac{1}{2}} = \frac{-\frac{630}{64}}{-\frac{1}{2}} = 19\frac{11}{16}.$

Art. **148.** If q is a proper fraction, $q - 1$ will be negative; $q^n - 1$ will also be negative, since any power of a proper fraction, if the index is greater than unity, is less than the fraction itself. Changing the signs of numerator and denominator, in the formula for S, we have $S = \frac{a(1 - q^n)}{1 - q}$, or $S = \frac{a - aq^n}{1 - q}$.

Now, as the powers of a fraction less than unity become less and less, the greater the index of the power, it follows that if n, the number of terms, is infinitely great, q^n must be infinitely small, that is, zero. In this case, $S = \frac{a - aq^n}{1 - q}$ will become $S = \frac{a - a \cdot 0}{1 - q} = \frac{a}{1 - q}$.

Since q is supposed to be a fraction, let it be represented by $\frac{m}{n}$, so that $q = \frac{m}{n}$. We shall then have $S = \frac{a}{1 - \frac{m}{n}} = \frac{na}{n - m}$.

The formula, therefore, for the sum of a decreasing progression by quotient, continued to infinity, is

$$S = \frac{n\,a}{n - m}.$$

Hence,

To find the sum of an infinite decreasing series in progression by quotient, multiply the first term by the denominator, and divide the product by the difference between the denominator and numerator of the ratio.

If q is a fraction whose numerator is 1, the formula for the sum of an infinite decreasing series, becomes $S = \dfrac{n\,a}{n - 1}$.

Ex. What is the sum of the series, 5, $\frac{10}{3}$, $\frac{20}{9}$, &c., continued to infinity.

In this example, $a = 5$, and $q = \frac{2}{3}$; therefore, $S = \dfrac{5 \,.\, 3}{3 - 2} =$ 15.

Art. **149**. From the formula for the sum of an infinite decreasing progression by quotient, may be deduced the rule given in arithmetic, for reducing *periodical fractions*, sometimes called *repeating* and *circulating decimals*, to vulgar fractions.

Let us take the decimal ·333 continued infinitely. This is a progression by quotient, in which the first term a is $\frac{3}{10}$, the second $\frac{3}{100}$, &c., the common ratio q being $\frac{1}{10}$. Hence, by substitution, the formula, $S = \dfrac{a\,n}{n - 1}$, becomes $S = \dfrac{\frac{3}{10} \,.\, 10}{10 - 1} = \frac{3}{9} = \frac{1}{3}$.

Again, in the fraction, ·0404 &c., $a = \frac{4}{100}$, and $q = \frac{1}{100}$; hence, $S = \dfrac{a\,n}{n - 1}$ becomes $S = \dfrac{\frac{4}{100} \,.\, 100}{100 - 1} = \frac{4}{99}$.

In like manner, the sum of ·296296 &c., $= \dfrac{\frac{296}{1000} \,.\, 1000}{1000 - 1} = \frac{296}{999}$.

Let us take the fraction ·428571428571 &c. Here $a = \frac{428571}{1000000}$, and as the second period is ·000000428571, $q = \frac{1}{1000000}$. Therefore $S = \dfrac{\frac{428571}{1000000} \cdot 1000000}{1000000 - 1} = \frac{428571}{999999}$, which reduced, is $\frac{3}{7}$.

Consequently we see, as in arithmetic, *that the repeating or circulating figures are to be made the numerator of a fraction, whose denominator is as many 9s as there are repeating figures, and then the resulting fraction is to be reduced to its lowest terms.*

If the repeating figures do not commence immediately after the decimal point, we have only to find the value of the repeating part, and add it to the decimal which precedes, reducing them both to the same denominator.

For example, ·5333 &c. $= \frac{5}{10} + \frac{3}{100} + \frac{3}{1000}$ &c., in which the first term of the progression is $\frac{3}{100}$, and the ratio $\frac{1}{10}$; in this case, the sum of the whole is $\frac{5}{10} + \dfrac{\frac{3}{100} \cdot 10}{9} = \frac{5}{10} + \frac{3}{90} =$ $\dfrac{45 + 3}{90} = \frac{48}{90} = \frac{8}{15}$.

Art. **150**. Suppose that a and l were given, and it were required to insert between them any number of terms, such that the whole should form a progression by quotient. The terms thus introduced are called *mean proportionals*, or *geometric means*.

Making an equation between the two values of S, given in the two formulæ, we have

$$\frac{a q^n - a}{q - 1} = \frac{q l - a}{q - 1};$$ from which we derive successively

$$a q^n - a = q l - a, \quad a q^n = q l, \quad a q^{n-1} = l,$$

$$q^{n-1} = \frac{l}{a}, \text{ and } q = \sqrt[n-1]{\frac{l}{a}}.$$

Now, if it be required to insert m terms between a and l, since there would be $m + 2$ terms in the whole, we put $m + 2$ instead of n, in the value of q, just found.

We have then

$$q=\sqrt[m+1]{\frac{l}{a}}.\quad \text{Hence,}$$

When any number of mean proportionals is to be inserted between two quantities, to find the common ratio, divide the greater quantity by the less, and extract the root of the quotient to the degree denoted by the number of terms to be inserted plus unity.

Knowing the common ratio, it is easy to write the progression, which is expressed in general terms as follows, viz:

$$a,\ a\sqrt[m+1]{\frac{l}{a}},\ a\sqrt[m+1]{\left(\frac{l}{a}\right)^2},\ a\sqrt[m+1]{\left(\frac{l}{a}\right)^3},\ldots\ldots,l.$$

Ex. Insert five mean proportionals between 2 and 128.

In this example the formula, $q=\sqrt[m+1]{\frac{l}{a}}$, becomes $q=\sqrt[6]{\frac{128}{2}}=\sqrt[6]{64}=2$; and the progression is 2, 4, 8, 16, 32, 64, 128.

It is manifest, that the same number of mean proportionals may be inserted between the terms of a progression by quotient taken two and two, and the result will be in progression.

Ex. Between every two adjacent terms of 3, 24, 192, insert two mean proportionals.

In this case, $q=\sqrt[3]{\frac{24}{3}}=\sqrt[3]{8}=2$; and the resulting progression is 3, 6, 12, 24, 48, 96, 192.

Art. **151**. In the formulæ already given, a, q and n were supposed to be known, and it was required to find l and S. But if any three of the five quantities, a, q, n, l and S, are known, the remaining two may be found.

There may be, therefore, ten different problems; but the student, at this stage of his progress, is capable of solving only four of them. Four of the remaining six will be solved under the

head of logarithms; but the two others give rise to equations too difficult of solution to be admitted into an elementary treatise.

Let the pupil solve the following problems.

1. Given a, q and n; find l and S.

Note. This question has already been solved, and the results may be assumed in solving those which follow.

2. Given a, n and l; find q and S.
3. Given q, n and l; find a and S.
4. Given q, n and S; find a and l.

SECTION LIII.

EXAMPLES IN PROGRESSION BY QUOTIENT.

Art. **152**. 1. Required the last term and the sum of the progression, 5, 10, 20, &c., the number of terms being 8.

2. What is the 5th term, and the sum of the first five terms of the progression, 1, $\frac{1}{3}$, $\frac{1}{9}$, &c.?

3. There are three numbers forming a geometrical progression, in which the mean is 10, and the sum of the extremes 52. Required the numbers.

Let $x =$ the ratio. Then $\frac{10}{x}$, 10, and $10x$ will represent the terms.

4. A gentleman, without reflecting upon the result, agreed to pay his gardener 1 dollar for the first month, two for the second, and so on, doubling his wages each month for a year. What would be the amount of the year's wages?

5. There are four numbers in progression by quotient; the sum of the first three is 130, and that of the last three is 390. Required the numbers.

Let $x =$ the first number, and $y =$ the common ratio.

Then, x, xy, xy^2, xy^3 will represent the numbers, and we have the equations,

$$(1) \quad x + xy + xy^2 = 130;$$
$$(2) \quad xy + xy^2 + xy^3 = 390.$$

Separating the first members into factors, we have

$$(3) \quad x(1 + y + y^2) = 130;$$
$$(4) \quad xy(1 + y + y^2) = 390.$$

Divide the 4th by the 3d; this gives

$$y = 3.$$

Substituting this value of y in the 3d, we have

$$x(1 + 3 + 9) = 130, \text{ or}$$
$$13x = 130; \text{ consequently,}$$
$$x = 10.$$

The numbers are, therefore, 10, 30, 90, 270.

6. There are five numbers in progression by quotient; the sum of the first four is 468, and that of the last four is 2340. What are these numbers?

7. Divide 217 into three parts which shall form a geometrical progression, such that the third term shall exceed the first by 168.

8. The sum of three numbers in progression by quotient is 104; and the mean term is to the sum of the extremes as 3 to 10. Required the numbers.

9. There are three numbers in progression by quotient, and the sum of the first and second is to the sum of the second and third as 1 to 2. What are these numbers?

10. There are three numbers in progression by difference, such that if the second power of the first be increased by 1, that of the second by 5, and that of the third by 41, the results will form a progression by quotient, in which the sum of the terms will be 130, and the sum of the extremes will be to the mean as 10 to 3. Required the numbers.

11. Find a mean proportional between 9 and 4.

12. Find a mean proooortional between 4 and 25.

13. Find a mean proportional between 7 and 9, carried to three decimals.

14. Find a mean proportional between 75 and 425, accurate to three decimals.

15. Find the sum of the series, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., carried to infinity.

16. Find the sum of the series, 4, $\frac{4}{3}$, $\frac{4}{9}$, &c., continued to infinity.

17. Find the sum of 7, $\frac{14}{3}$, $\frac{28}{9}$, &c., continued to infinity.

18. What is the sum of 81, 9, 1, $\frac{1}{9}$, &c., continued to infinity?

19. Insert three mean proportionals between 2 and 162.

20. Insert two mean proportionals between 5 and 1080.

21. Insert a mean proportional between every two adjacent terms of the progression, 3, 75, 1875, 46875.

22. Insert two mean proportionals between every two adjacent terms of the series, 1, 8, 64, 512.

SECTION LIV.

EXERCISES IN EQUATIONS OF THE SECOND DEGREE.

Art. **153**. Solve the following equations.

1. Given $\sqrt{6+3x}=6$; to find x.

Squaring both members, we have

$$6+3x=36; \text{ hence,}$$

$$x=10. \quad \text{Ans.}$$

2. Given $(16+x^2)^{\frac{1}{2}}-x=2$; to find x.

By transposition,

$(16+x^2)^{\frac{1}{2}}=x+2$; squaring both members,

$16+x^2=x^2+4x+4$; transposing,

$x^2-x^2-4x=4-16$; reducing, changing the signs and dividing,

$x=3$. Ans.

3. Given $\dfrac{\sqrt{x}+28}{\sqrt{x}+4} = \dfrac{\sqrt{x}+38}{\sqrt{x}+6}$; to find x.

Clearing the equation of fractions and reducing,

$x + 34\sqrt{x} + 168 = x + 42\sqrt{x} + 152$; transposing,

$x + 34\sqrt{x} - x - 42\sqrt{x} = 152 - 168$; reducing,

$-8\sqrt{x} = -16$; changing the signs and dividing by 8,

$\sqrt{x} = 2$; squaring,

$x = 4$. Ans.

4. Given $\sqrt[2m]{x^2 + 5ax + b^2} = \sqrt[m]{a + x}$; to find x.

Raising both members to the mth power,

$\sqrt{x^2 + 5ax + b^2} = a + x$; squaring,

$x^2 + 5ax + b^2 = a^2 + 2ax + x^2$; transposing and reducing,

$3ax = a^2 - b^2$; dividing by $3a$,

$x = \dfrac{a^2 - b^2}{3a}$. Ans.

5. Given $(x+6)^{\frac{1}{4}} = (x-6)^{\frac{1}{2}}$; to find x.

Raising both members to the 4th power,

$x + 6 = (x-6)^2$, or

$x + 6 = x^2 - 12x + 36$; inverting the members,

$x^2 - 12x + 36 = x + 6$; transposing and reducing,

$x^2 - 13x = -30$.

Now, by substituting in formula 4th, Art. **96**,

$x = \frac{13}{2} \pm \sqrt{-30 + \frac{169}{4}} = \frac{13}{2} \pm \frac{7}{2}$. Hence,

$x = 10$, or $x = 3$. Ans.

We see, from the preceding examples, that an equation containing radical quantities, may generally be freed from them, by raising both members to the power denoted by the degree of the radical, the operation being repeated, if necessary. When some of the terms do not contain radicals, it is usually best, in the first place, to make them constitute one member, and the remaining terms the other.

Many of the problems in this and the following section will give several answers each, if the double sign $\pm$ be prefixed to roots of an even degree.

Find x in the following equations.

6. $\sqrt[4]{x-1}+4=9.$
7. $x+(x+6)^{\frac{1}{2}}=2+3(x+6)^{\frac{1}{2}}.$
8. $\sqrt{12+x}=\sqrt{x}+2.$
9. $\sqrt{x}-2=4-3\sqrt{x}.$
10. $\sqrt{x+7}=\sqrt{x}+1.$
11. $(x-32)^{\frac{1}{2}}=16-x^{\frac{1}{2}}.$

The mode pursued in the preceding questions, frequently leads to equations of so elevated a degree, that their solution would be too difficult for an elementary work. Other expedients, therefore, are often to be preferred.

Whenever an equation can be reduced to the form of $x^{2m}\pm px^m=\pm q$, that is, to an equation, in which the unknown quantity is found in two terms only, and has an exponent in one of them double its exponent in the other, it may be solved after the manner of affected equations of the second degree.

12. Given $x^4+6x^2=135$; to find x.

First consider x^2 as the unknown quantity, and make the first member a perfect square,

$x^4+6x^2+9=144$; extracting the square root,
$x^2+3=\pm 12$; transposing and reducing,
$x^2=9$ or -15; taking the square root of this,
$x=\pm 3$, or $x=\pm\sqrt{-15}.$

Hence, $x=3$, $x=-3$, $x=\sqrt{-15}$, or $x=-\sqrt{-15}.$

This question might have been solved by means of formula 1st, Art. **96**.

Thus, $x^2=-3\pm\sqrt{135+\frac{36}{4}}=-3\pm 12=9$ or $=-15$; then $x=\pm 3$, or $x=\pm\sqrt{-15}.$

13. Given $x+4\sqrt{x}=32$; or what is the same, $x+4x^{\frac{1}{2}}=32$; to find x.

In this equation consider $\sqrt{x}$ as the unknown quantity. By the formula 1st, Art. **96**, we have

$$\sqrt{x}=-2\pm\sqrt{32+4};$$

$\sqrt{x}=4$, or $\sqrt{x}=-8$; squaring both equations,
$x=16$, or $x=64$. Ans.

14. Given $2\sqrt{x}+3\sqrt[4]{x}=27$; to find x.

Dividing by 2,

$$\sqrt{x}+\tfrac{3}{2}\sqrt[4]{x}=\tfrac{27}{2}.$$

Referring to the formula, considering $\sqrt[4]{x}$ as the unknown quantity,

$\sqrt[4]{x}=-\frac{3}{4}\pm\sqrt{\frac{27}{2}+\frac{9}{16}}$; hence,
$\sqrt[4]{x}=3$, or $\sqrt[4]{x}=-\frac{9}{2}$; taking the 4th power of both equations,

$x=81$, or $x=\frac{6561}{16}$. Ans.

15. Given $x^{\frac{1}{3}}+10=5x^{\frac{1}{6}}+4$; to find x.

Transposing and reducing,

$x^{\frac{1}{3}}-5x^{\frac{1}{6}}=-6$; completing the square,

$x^{\frac{1}{3}}-5x^{\frac{1}{6}}+\frac{25}{4}=-6+\frac{25}{4}$; or, reducing the second member,

$x^{\frac{1}{3}}-5x^{\frac{1}{6}}+\frac{25}{4}=\frac{1}{4}$; taking the square root,

$x^{\frac{1}{6}}-\frac{5}{2}=\pm\frac{1}{2}$; transposing and reducing,

$x^{\frac{1}{6}}=3$, or $x^{\frac{1}{6}}=2$; raising both equations to the 6th power,

$x=729$, or $x=64$. Ans.

If we had substituted in formula 4th, Art. **96**, the operation would have been shorter.

16. Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$; to find x.

Taking the roots of the perfect squares and placing them before the radical sign,

$$x^2\sqrt{x}+x\sqrt{x}=6\sqrt{x}; \text{ dividing by } \sqrt{x},$$
$$x^2+x=6.$$

This may now be solved like any affected equation; and the following equations may be solved like the preceding.

17. $2x^4+8x^2=24.$

18. $x^6-8x^3-513=0.$

19. $\dfrac{3\sqrt{x}}{5}-2-\dfrac{x-5}{20}=0.$

20. $x^{\frac{4}{3}}+7x^{\frac{2}{3}}-44=0.$

21. $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39.$

22. $3x^6+42x^3=3321.$

23. $x^{\frac{3}{2}}+x^{\frac{5}{2}}=6x^{\frac{1}{2}}.$

24. $x-4x^{\frac{1}{2}}=45.$

25. $4x^{\frac{2}{3}}=7x^{\frac{1}{3}}-6.$

26. $5x^{\frac{4}{3}}-3x^{\frac{2}{3}}=4x^{\frac{2}{3}}+342.$

27. $3x-4\sqrt{x}=240.$

28. $3\sqrt{x}+7\sqrt[4]{x}=48.$

29. $\dfrac{\sqrt{4x}+2}{4+\sqrt{x}}=\dfrac{4-\sqrt{x}}{\sqrt{x}}.$

30. $\sqrt{x^3}-2\sqrt{x}-x=0.$

31. $ax^{\frac{3}{2}}+bx^{\frac{3}{4}}=c.$

32. Given $x+5=\sqrt{x+5}+6$; to find x.

By transposition,

$$x+5-\sqrt{x+5}=6.$$

Considering $\sqrt{x+5}$ as the unknown quantity, and completing the square,

$x+5-\sqrt{x+5}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$; taking the root,

$\sqrt{x+5}-\frac{1}{2}=\pm\frac{5}{2}$; transposing and reducing,

$\sqrt{x+5}=3$, or $\sqrt{x+5}=-2$; squaring both equations,

$x+5=9$, or $x+5=4$; transposing and reducing,
$x=4$, or $x=-1$. Ans.

Another method.

Resume the equation, $x+5-\sqrt{x+5}=6$.

Substitute some letter, as y, instead of $\sqrt{x+5}$; then we have

$\sqrt{x+5}=y$, and consequently $x+5=y^2$; hence, the equation becomes

$y^2-y=6$. This gives, by the formula,
$y=\frac{1}{2}\pm\sqrt{6+\frac{1}{4}}=3$, or $y=-2$; therefore,
$y^2=9$; or $y^2=4$.

But $y^2=x+5$; hence,

$x+5=9$; or $x+5=4$; transposing and reducing,
$x=4$; or $x=-1$. Ans.

The latter method of solution is preferable, as it saves the necessity of repeatedly writing a polynomial.

33. Given $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$; to find x.

Adding 9 to each member,

$2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$.

Let $y=\sqrt{2x^2+3x+9}$; then $y^2=2x^2+3x+9$; hence,

$y^2-5y=6$; then, by formula 2d, Art. **96**,

$y=\frac{5}{2}\pm\sqrt{6+\frac{25}{4}}=6$, or $y=-1$; hence,
$y^2=36$, or $y^2=1$. Taking the 1st value of y^2,
$2x^2+3x+9=36$;
$x^2+\frac{3}{2}x=\frac{27}{2}$;
$x=-\frac{3}{4}\pm\sqrt{\frac{27}{2}+\frac{9}{16}}$;
$x=3$; or $x=-\frac{9}{2}$.

Taking the other value of y^2, viz: 1,

$$2x^2+3x+9=1;$$
$$x^2+\tfrac{3}{2}x=-4;$$
$$x=-\tfrac{3}{4}\pm\sqrt{-4+\tfrac{9}{16}};$$
$$x=-\tfrac{3}{4}\pm\sqrt{-\tfrac{55}{16}};$$
$$x=\frac{-3\pm\sqrt{-55}}{4}.$$

Solve the following equations.

34. $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}.$

35. $x+\sqrt{x+6}=2+3\sqrt{x+6}.$

36. $x^2-2x+6\sqrt{x^2-2x+5}=11.$

37. $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2.$

38. $(x-5)^3-3(x-5)^{\frac{3}{2}}=40.$

39. $\dfrac{\sqrt{x^2+x+6}}{3}=\dfrac{18-(\frac{4}{3}\sqrt{x^2+x+6}-2)}{\sqrt{x^2+x+6}}.$

40. $x^2-x+5\sqrt{2x^2-5x+6}=\dfrac{3x+33}{2}.$

SECTION LV.

EXERCISES IN EQUATIONS OF THE SECOND DEGREE WITH TWO UNKNOWN QUANTITIES.

Art. **154.** 1. Given $\left\{\begin{array}{l} x+y=3x-3y \\ x^2-y^2=12 \end{array}\right\}$; to find x and y.

From the 1st equation,

$x=2y$; substitute this value of y in the 2d equation,

$4y^2-y^2=12;$

$3y^2=12;$

$y^2=4;$

$y=\pm 2.$

Hence, $x = 2y = \pm 4$.

In the preceding question, the value of one unknown quantity was found in one equation, and substituted in the other; and in this way the solution can be effected, when one of the given equations is of the first degree, and the other of the second.

Find x and y in the following equations.

2. $\begin{cases} \dfrac{4x+2y}{3} = 6. \\ 5xy = 50. \end{cases}$

3. $\begin{cases} \dfrac{x}{y^2} = 2. \\ x - y = 15. \end{cases}$

4. $\begin{cases} x + 4y = 14. \\ y^2 - 2y + 4x = 11. \end{cases}$

5. $\begin{cases} \dfrac{x+y}{y} = 3. \\ x^2 + 3y^2 = 28. \end{cases}$

When both equations are above the first degree, different expedients are to be adopted, which will be best learned from examples.

6. $\begin{cases} (1) \quad xy = 50. \\ (2) \quad x^2 + y^2 = 125. \end{cases}$

Adding twice the 1st to the 2d,

(3) $x^2 + 2xy + y^2 = 225$; taking the square root,

(4) $x + y = \pm 15$.

Subtracting twice the 1st from the 2d,

(5) $x^2 - 2xy + y^2 = 25$; taking the square root,

(6) $x - y = \pm 5$.

Adding the 4th and 6th, and dividing by 2,

$$x = \pm 10.$$

Subtracting the 6th from the 4th, and dividing by 2,

$$y = \pm 5.$$

By taking all the possible combinations of the signs in the 2d members of the 4th and 6th, we have

$x = 10$, and $y = 5$; or,
$x = -10$, and $y = -5$; or,
$x = 5$, and $y = 10$; or,
$x = -5$, and $y = -10$.

7. $\begin{cases} (1) \quad x^2 + xy = 12. \\ (2) \quad xy + y^2 = 24. \end{cases}$

Adding the 1st and 2d,

(3) $x^2 + 2xy + y^2 = 36$; taking the root,
(4) $x + y = \pm 6$.

But, $x^2 + xy = 12$ is the same as

$(x + y)x = 12$; substituting in this ± 6 instead of $x + y$,

$\pm 6x = 12$;
$x = \pm 2$.

Substituting this value of x in the 4th,

$\pm 2 + y = \pm 6$;
$y = \pm 6 \mp 2$, or,
$y = \pm 4$.

In the last equation but one, the upper signs correspond, as also do the lower.

8. $\begin{cases} (1) \quad x + y = s. \\ (2) \quad xy = a^2. \end{cases}$

Squaring the 1st,

(3) $x^2 + 2xy + y^2 = s^2$; subtracting from this 4 times the 2d,

(4) $x^2 - 2xy + y^2 = s^2 - 4a^2$; taking the square root,

(5) $x - y = \pm\sqrt{s^2 - 4a^2}$.

Adding the 1st and 5th, and dividing by 2,

$$x = \frac{s \pm \sqrt{s^2 - 4a^2}}{2}.$$

Subtracting the 5th from the 1st, and dividing by 2,

$$y = \frac{s \mp \sqrt{s^2 - 4a^2}}{2}.$$

9. $\begin{cases} (1) & x^2 y + x y^2 = 180. \\ (2) & x^3 + y^3 = 189. \end{cases}$

Adding 3 times the 1st to the 2d,

(3) $x^3 + 3x^2 y + 3xy^2 + y^3 = 729$; taking the 3d root,

(4) $x + y = 9$.

But the first member of the 1st is the same as $xy(x+y)$; substituting 9 for $x+y$, the 1st becomes

(5) $9xy = 180$; hence,

(6) $xy = 20$.

Squaring the 4th,

(7) $x^2 + 2xy + y^2 = 81$; subtracting from this 4 times the 6th,

(8) $x^2 - 2xy + y^2 = 1$; taking the 2d root,

(9) $x - y = \pm 1$; adding the 4th and 9th and dividing by 2,

$$x = \frac{9 \pm 1}{2} = 5, \text{ or } 4.$$

Subtracting the 9th from the 4th, and dividing by 2,

$$y = \frac{9 \mp 1}{2} = 4, \text{ or } 5.$$

10. $\begin{cases} (1) & x^4 - y^4 = 1776. \\ (2) & x^2 - y^2 = 24. \end{cases}$

Divide the 1st by the 2d,

(3) $x^2 + y^2 = 74$; adding the 2d and 3d, and dividing by 2,

$x^2 = 49$; hence,

$x = \pm 7$.

Subtracting the 2d from the 3d, and dividing by 2,

$y^2 = 25$; hence,

$y = \pm 5$.

$y = -\frac{9}{8} \pm \sqrt{25 + \frac{81}{64}} = -\frac{9}{8} \pm \frac{41}{8}$;
$y = 4$, or $-6\frac{1}{4}$. Consequently,
$x = \frac{9y}{4} = 9$, or $-14\frac{1}{16}$.

26. $\begin{cases} (1) & xy = 24. \\ (2) & x^3 - y^3 : (x - y)^3 = 19 : 1. \end{cases}$

Dividing both terms of the 1st ratio of 2d by $x - y$, Prop. 6th,

(3) $x^2 + xy + y^2 : (x - y)^2 = 19 : 1$, or,
(4) $x^2 + xy + y^2 : x^2 - 2xy + y^2 = 19 : 1$; hence, Prop. 10th,
(5) $3xy : 18 = x^2 - 2xy + y^2 : 1$; substituting 24 for xy in the 1st term,
(6) $72 : 18 = (x - y)^2 : 1$; hence, Prop. 6th,
(7) $4 : 1 = (x - y)^2 : 1$; and, Prop. 14th,
(8) $(x - y)^2 = 4$; from which,
(9) $x - y = \pm 2$.

Adding 4 times the 1st to the square of the 9th,

(10) $x^2 + 2xy + y^2 = 100$; extracting the root,
(11) $x + y = \pm 10$.

Adding the 9th to the 11th, and dividing by 2,

$$x = \pm 6.$$

Subtracting the 9th from the 11th, and dividing by 2,

$$y = \pm 4.$$

27. $\begin{cases} x : y = 5 : 4. \\ x + 5 : y - 1 = 5 : 3. \end{cases}$

28. $\begin{cases} x : y = 5 : 3. \\ x^2 - y^2 : (x - y)^2 = x - 1 : 1. \end{cases}$

29. $\begin{cases} xy = 48. \\ x^3 - y^3 : (x - y)^3 = 37 : 1. \end{cases}$

30. $\begin{cases} x^2 : y^2 = 49 : 36. \\ 2x - y : x + 6 = 16 : 20. \end{cases}$

31. $\begin{cases} y : 55 - y = x : 10 - x. \\ 2y - 41 = (11 - x)^2. \end{cases}$

32. $\begin{cases} x^2+y^2 : x^2-y^2 = 17:8. \\ x\,y^2 = 45. \end{cases}$

When two equations of the second degree, containing two unknown quantities, are homogeneous with respect to these quantities, that is, when each unknown term contains either the square of one of the unknown quantities, or the product of both, the solution can be effected by substituting, for one of the unknown quantities, the product of the other by a new unknown quantity.

33. $\begin{cases} (1) \quad 2x^2 - xy = 6. \\ (2) \quad 2y^2 + 3xy = 8. \end{cases}$

Let $x = zy$.

Substituting this value of x in the given equations,

(3) $2z^2y^2 - zy^2 = 6.$

(4) $2y^2 + 3zy^2 = 8.$ From the 3d,

(5) $y^2 = \dfrac{6}{2z^2 - z}$; from the 4th,

(6) $y^2 = \dfrac{8}{2+3z}$. Hence,

$\dfrac{8}{2+3z} = \dfrac{6}{2z^2 - z}$. Solving this equation,

$$z = \tfrac{13}{16} \pm \sqrt{\tfrac{6}{8} + (\tfrac{13}{16})^2} = \tfrac{13}{16} \pm \sqrt{\tfrac{361}{256}} = 2, \text{ or } -\tfrac{3}{8};$$

hence,

$y^2 = \dfrac{8}{2+6} = 1$; and,

$y = \pm 1$; or,

$y^2 = \dfrac{8}{2 - \frac{9}{8}} = \frac{64}{7}$; and,

$y = \pm \dfrac{8}{\sqrt{7}}$.

Therefore, $x = zy = \pm 1 \,.\, 2 = \pm 2$; or,

$$x = (-\tfrac{3}{8}) \cdot \left(\pm \frac{8}{\sqrt{7}}\right) = \mp \frac{3}{\sqrt{7}}.$$

Solve the following equations in a similar manner.

34. $\begin{cases} x^2 + x\,y = 12. \\ x\,y - 2\,y^2 = 1. \end{cases}$

35. $\begin{cases} 3\,x^2 - 3\,x\,y + y^2 = 21. \\ 2\,x\,y = 3\,y^2 + x^2 - 19. \end{cases}$

36. $\begin{cases} 4\,x^2 = 3\,x\,y - 2. \\ x^2 + y^2 = 5. \end{cases}$

37. $\begin{cases} 5\,x^2 - 3\,x\,y = 56. \\ 5\,y^2 + x\,y = 28. \end{cases}$

SECTION LVI.

LOGARITHMS.

Art. **155.** We have already seen, that two different powers of the same quantity are multiplied together by adding the exponents, and divided one by the other by subtracting the exponent of the divisor from that of the dividend; also, that any power of a quantity is found by multiplying the exponent, and any root is found by dividing the exponent, by the number expressing the degree of the power or root.

Let us construct a table of powers of any number, as 2, for example, placing the powers in one column and the exponents in another.

Powers.	Exponents.	Powers.	Exponents.	Powers.	Exponents.
1	0	256	8	65536	16
2	1	512	9	131072	17
4	2	1024	10	262144	18
8	3	2048	11	524288	19
16	4	4096	12	1048576	20
32	5	8192	13	2097152	21
64	6	16384	14	4194304	22
128	7	32768	15	8388608	23

Suppose now it were required to multiply 128 by 1024.

Looking in the table, we find against 128 the exponent 7, and against 1024, the exponent 10; these exponents being added give 17. We now find 17 in the column of exponents, and against it, in the column of powers, we find 131072, which is the product of 128 by 1024; that is, $128 \cdot 1024 = 2^7 \cdot 2^{10} = 2^{17} = 131072$.

Divide 2097152 by 64.

Looking in the table, we find 21 for the exponent corresponding to the dividend, and 6 for that corresponding to the divisor; subtracting the latter from the former, we have 15 for the exponent corresponding to the quotient; we now find 15 among the exponents, and against it stands 32768, which is the quotient required. That is, $\frac{2097152}{64} = \frac{2^{21}}{2^6} = 2^{15} = 32768$.

Find the third power of 64

The exponent against 64 is 6, which multiplied by 3 gives 18; against the exponent 18 we find 262144, which is the power required. That is, $(64)^3 = (2^6)^3 = 2^{18} = 262144$.

Find the fifth root of 32768.

Against 32768 we find the exponent 15, which divided by 5, gives 3; against the exponent 3 stands 8, which is the root required. That is, $(32768)^{\frac{1}{5}} = (2^{15})^{\frac{1}{5}} = 2^{\frac{15}{5}} = 2^3 = 8$.

Let the learner find the answers to the following questions by means of the table.

1. Multiply 16 by 128.
2. Multiply 1024 by 64.
3. Multiply 512 by 2048.
4. Multiply 256 by 16384
5. Multiply 256 by 512.
6. Multiply 32768 by 128.
7. Divide 2097152 by 65536.
8. Divide 32768 by 1024.
9. Divide 262144 by 16384.

10. Divide 524288 by 512.
11. Divide 4096 by 256.
12. Divide 8388608 by 131072.
13. Find the 3d power of 32.
14. Find the 2d power of 128.
15. Find the 4th power of 16.
16. Find the 2d power of 1024.
17. Find the 4th power of 32.
18. Find the 5th power of 16.
19. Find the 2d root of 1024.
20. Find the 3d root of 512.
21. Find the 6th root of 262144.
22. Find the 4th root of 65536.
23. Find the 5th root of 32768.
24. Find the 7th root of 2097152.

Art. **156**. The number 2, which is raised to the several powers in the preceding table, is called the *base* of the table; and the exponents of the several powers, are called *logarithms* of the numbers to which those powers are equal. Thus, if 2 is the base of the table, the logarithm of 256 would be 8, and that of 16384 would be 14.

A table might be made, having for its base 3, 5, or any number except 1. Unity would not answer for a base, because all the powers as well as all the roots of 1 are 1.

Tables of logarithms in common use, are constructed upon the number 10 as a base.

Hence,

The common logarithm of a number, is the exponent of the power to which 10 *must be raised, in order to produce that number.*

Thus, 3 is the logarithm of 1000, because $10^3 = 1000$; and 0·5 is the logarithm of 3·162277, because $10^{0\cdot5} = \sqrt{10} =$ 3·162277, nearly.

Remark. We shall hereafter use the expression *log.* for the words "logarithm of."

Now $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, &c. Therefore, log. $1 = 0$, log. $10 = 1$, log. $100 = 2$, log. $1000 = 3$, log. $10000 = 4$, &c.

Again, $10^{-1} = \frac{1}{10} = \cdot 1$, $10^{-2} = \frac{1}{100} = \cdot 01$, $10^{-3} = \frac{1}{1000} = \cdot 001$, $10^{-4} = \frac{1}{10000} = \cdot 0001$, Art. **133**. Therefore, log. $\cdot 1 = -1$, log. $\cdot 01 = -2$, log. $\cdot 001 = -3$, log. $\cdot 0001 = -4$.

Hence, the logarithm of a number between 1 and 10 must be a fraction, that of a number between 10 and 100, 1 + a fraction, that of a number between 100 and 1000, 2 + a fraction, and so on.

It also appears, that the logarithm of a fraction less than unity, must be negative, and that the logarithms of intermediate numbers between ·1 and ·01, ·01 and ·001, ·001 and ·0001, will consist of whole numbers and fractions.

Art. **157**. To form a conception of the construction of logarithms, let us place some of the powers of 10 in one line, and their exponents or logarithms in another beneath. Thus,

1 or 10^0,	10,	100,	1000,	10000,	100000.
0,	1,	2,	3,	4,	5.

If we examine these two series, we shall perceive that the former is a progression by quotient, and the latter a progression by difference.

Now if we insert between the terms of the former series taken two and two, any number of mean proportionals, Art. **150**, and between the terms of the latter taken two and two, the same number of mean differentials, Art. **144**, the terms of the latter result will be the logarithms of the corresponding terms of the former result.

Thus, the mean proportional between 1 and $10 = \sqrt{1 . 10} = 3\cdot 162277$; and the mean differential between 0 and $1 = \frac{1-0}{2} = \frac{1}{2} = \cdot 5$. Then, log. $3\cdot 162277 = \cdot 5$.

If however we were to insert a very great number of mean proportionals between 1 and 10, we should find among them terms which differ very little from 2, 3, 4, 5, 6, 7, 8 and 9, and which therefore might be considered equal to these numbers.

Indeed this difference would be less, in proportion as the number of means inserted was greater, so that the approximation might be carried to any degree of accuracy.

If we insert now between 0 and 1 a number of mean differentials, equal to the number of mean proportionals inserted between 1 and 10, the terms of the result would be the logarithms of the terms of the series previously found, and those corresponding to 2, 3, 4, &c. would be the logarithms of these numbers.

This process which we have given, is designed to show the learner the possibility of constructing logarithms, rather than as a mode which can conveniently be reduced to practice.

The methods by which logarithms are actually calculated, are in general very different from that given above, and are too complicated to be introduced into an elementary treatise.

Suppose then that we have a table containing the logarithms of all the natural numbers, 1, 2, 3, &c., to any definite extent. In such a table the logarithm of 2, for example, is ·30103; that is, $10^{\frac{30103}{100000}} = 2$. This signifies, that, if 10 were raised to the 30103d power, and then the 100000th root were extracted, the result would be very nearly 2.

Art. **158**. Since logarithms are exponents, they are subject to the rules previously given for exponents. Hence,

1. *To multiply numbers together, add their logarithms; the sum will be the logarithm of the product.*

2. *To divide one number by another, subtract the logarithm of the divisor from that of the dividend; the difference will be the logarithm of the quotient.*

3. *To raise a number to any power, multiply its logarithm by the number expressing the degree of the power; the product will be the logarithm of the power required.*

4. *To extract any root of a number, divide its logarithm by the number expressing the degree of the root, or, what amounts to the same, multiply its logarithm by the fractional exponent which indicates the root; the result will be the logarithm of the root required.*

5. *Since a fraction expresses division, the logarithm of a fraction is found, by subtracting the logarithm of the denominator from that of the numerator.*

6. *The logarithm of either extreme of a proportion is found by adding the logarithms of the means, and from the sum subtracting the logarithm of the other extreme; also the logarithm of either mean is found, by subtracting that of the other mean from the sum of the logarithms of the extremes.*

Art. **159**. In constructing a table of logarithms, it is only necessary, in the first place, to calculate those of the prime numbers; from these the logarithms of all compound numbers may be deduced by addition and multiplication.

Thus, the logarithms of 2 and 3 being found, by adding them we have that of 6. In fact, the log. $2 = 0{\cdot}3010300$, and log. $3 = 0{\cdot}4771213$; hence, log. $6 = 0{\cdot}3010300 + 0{\cdot}4771213 = 0{\cdot}7781513$.

Again, $2 \times$ log. $2 = 0{\cdot}6020600 =$ log. 4, and $3 \times$ log. $2 = 0{\cdot}9030900 =$ log. 8, &c.

Hence, from the logarithms of 2 and 3, we easily obtain those of all the powers of these numbers, as well as those of all the combinations of these powers.

From the mode of performing multiplication by means of logarithms, it follows that the logarithms of those numbers which are 10, 100, 1000, &c. times the one of the other, must have their decimal parts the same, and can differ only with regard to their integral parts.

Thus, the logarithm of 2 being $0{\cdot}3010300$, the logarithm of $10 \,.\, 2$ or $20 =$ log. $10 +$ log. $2 = 1 + 0{\cdot}3010300 = 1{\cdot}3010300$; log. $200 =$ log. $100 +$ log. $2 = 2{\cdot}3010300$; log. $2000 =$ log. $1000 +$ log. $2 = 3{\cdot}3010300$. In like manner, the log. 3 being $0{\cdot}4771213$, we have log. $30 = 1{\cdot}4771213$, log. $300 = 2{\cdot}4771213$, log. $3000 = 3{\cdot}4771213$, &c.

Again the logarithm of 371250 being $5{\cdot}5696665$, we have

$$\text{log. } 37125 = \text{log. } \left(\tfrac{371250}{10}\right) = 4{\cdot}5696665,$$
$$\text{log. } 3712{\cdot}5 = \text{log. } \left(\tfrac{37125}{10}\right) = 3{\cdot}5696665,$$

$$\begin{aligned}
\log.\ 371{\cdot}25 &= \log.\ \left(\tfrac{3712{\cdot}5}{10}\right) &&= 2{\cdot}5696665,\\
\log.\ 37{\cdot}125 &= \log.\ \left(\tfrac{371{\cdot}25}{10}\right) &&= 1{\cdot}5696665,\\
\log.\ 3{\cdot}7125 &= \log.\ \left(\tfrac{37{\cdot}125}{10}\right) &&= 0{\cdot}5696665,\\
\log.\ {\cdot}37125 &= \log.\ \left(\tfrac{3{\cdot}7125}{10}\right) &&= -1{\cdot}5696665,\\
\log.\ {\cdot}037125 &= \log.\ \left(\tfrac{{\cdot}37125}{10}\right) &&= -2{\cdot}5696665,\\
\log.\ {\cdot}0037125 &= \log.\ \left(\tfrac{{\cdot}037125}{10}\right) &&= -3{\cdot}5696665,\\
\log.\ {\cdot}00037125 &= \log.\ \left(\tfrac{{\cdot}0037125}{10}\right) &&= -4{\cdot}5696665.
\end{aligned}$$

In dividing by 10, in each instance, we have subtracted the logarithm of 10, which is 1, from the logarithm of the dividend. In the last four examples, the subtraction is represented only, the decimal part being positive.

Art. **160.** We have before shown, that the logarithms of fractions less than unity, are negative; as represented above, however, the integral part alone is negative. But the negative part being greater than the positive, the expression as a whole is still negative. Since negative logarithms do not occur in the tables, we use the logarithms of decimals in the form given; and, to distinguish them from logarithms wholly negative, we place the minus sign over the integral part. Thus, log. $\cdot 0037125 = \bar{3}\cdot 5696665$.

The integral part of a logarithm is called its *characteristic*, because it always determines the order of units, expressed by the first significant figure of the corresponding number.

From what precedes we see, that,

The characteristic if positive, is always one less than the number of integral figures in the corresponding number; but, if the characteristic is negative, it is always equal to the number of places by which the first significant figure is removed to the right of the decimal point.

Thus, if 2 be the characteristic, there would be three figures, preceding the decimal point in the corresponding number; but if the characteristic be $\bar{1}$, the first figure of the number would be tenths, if it be $\bar{2}$, the first figure would be hundredths.

In logarithmic tables the characteristic is usually omitted, since we can never be at a loss to determine it, and since the

same decimal part of a logarithm, corresponds to several different numbers, composed of the same figures, but in which the figures express different orders of units.

SECTION LVII.

USE OF THE TABLES IN FINDING THE LOGARITHMS OF GIVEN NUMBERS, AND THE REVERSE.

Art. **161**. Logarithmic tables are usually accompanied with directions for using them, which are somewhat different in different works, according to the arrangement and extent of the tables. The principle, however, is in all cases the same.

In some tables, logarithms are carried only to five, in others to six, and in others to seven decimal places.

The student is supposed to be provided with a table of logarithms carried to seven decimals, extending to the number 10000. If, however, his tables are carried only to five or six decimals, he may disregard the last two, or the last decimal, in the logarithms which occur below.

Art. **162**. *To find from the tables the logarithm of a given number.*

If the number consists of less than four figures, whether it be integral, mixed or decimal, find the figures in the left hand column marked N. or Number, and, on the same horizontal line, in the next column to the right, will be found the decimal part of its logarithm, to which prefix the proper characteristic. In this, and all other cases, zeros to the right or left of the number will have no effect on the decimal part of the logarithm. Thus, log. $37 = 1{\cdot}5682017$; log. $3700 = 3{\cdot}5682017$; log. $385 = 2{\cdot}5854607$; log. $385000 = 5{\cdot}5854607$; log. $2{\cdot}57 = 0{\cdot}4099331$; log. $\cdot0573 = \bar{2}{\cdot}7581546$.

To find the logarithm of a number consisting of four figures, look for the first three figures in the left hand column, and the

fourth at the top of the page; then, on the same horizontal line with the first three, and beneath the fourth, that is, in the same vertical line with it, will be found the decimal of the logarithm, to which prefix its proper characteristic. Thus, log. 4796 = 3·6808792; log. ·03745 = $\bar{2}$·5734518*.

When the number contains more than four figures, find the decimal part of the logarithm of the first four, as already directed; then consider the remaining figures of the number as a fraction, placing a decimal point before them; multiply the difference between the logarithm already found and the next greater by this fraction; and, rejecting as many figures on the right as there are decimals in the multiplier, add the product to the decimal of the logarithm corresponding to the first four figures, remembering to place the right hand figures of the decimals to be added under each other; prefix the appropriate characteristic, and the result will be the logarithm sought.

For example, in finding the logarithm of 3745126, we take the decimal part of the logarithm corresponding to 3745, and add to it ·126 of the difference between that logarithm and the next greater, and to the result prefix 6 as a characteristic.

The reason of this method of proceeding will be seen as follows. The decimal logarithm of 3745000 is ·5734518; the next greater decimal logarithm, corresponding to 3746000, is ·5735678. The difference between these numbers is 1000, and the difference between their logarithms is ·0001160; moreover, 3745126 exceeds 3745000 by 126. Wherefore, if, when the number increases 1000, the logarithm increases ·0001160, when the number increases 126, the logarithm should increase $\frac{126}{1000}$, = ·126, of ·0001160, which is ·0000146160; retaining only seven decimals, we have ·0000146, which added to ·5734518, gives

* In the more extended tables, as those of Callet, four figures of the number are found in the left hand column, and the fifth at the top. Moreover, proportional parts of the differences, are placed on the right hand side of the page.

·5734664; to this sum prefix the characteristic 6, and we have log. 3745126 = 6·5734664.

The result would evidently have been the same, if we had disregarded the rank of the decimals in the difference of the logarithms, multiplied this difference by ·126, rejected the three right hand figures of the product, and added the reserved part of the product to 6·5734518, placing the right hand figure under the 8.

In like manner, we shall find log. 327983 = 5·5158514; also, log. ·0379426 = $\bar{2}$·5791271.

Remark. This mode of finding the logarithms of large numbers, as well as that to be given for finding numbers corresponding to given logarithms, supposes that logarithms increase in the same ratio as the numbers themselves, which, though not strictly true, is nearer the truth, the greater the numbers and the less their difference, and gives results sufficiently accurate for most practical purposes.

Let the learner find from his tables the logarithms of the following numbers.

1. 1273.	6. 12710·63.
2. 57293.	7. 2·74967.
3. ·01273.	8. 333·333
4. ·00279.	9. 435·501.
5. 327496.	10. 111·3734

Art. **163.** *To find the number corresponding to any given logarithm.*

Look for the decimal part of the logarithm in the table, and if it be found exactly, the first three figures of the number will be found in the left hand column, marked N., in the same horizontal line with the logarithm, and the fourth at the top, directly above the logarithm, the rank of the figures being shown by the characteristic.

Thus, 3·5860244 being the given logarithm, the corresponding number is 3855. Had the characteristic been 1, the num-

ber would have been 38·55; had it been 6, the number would have been 3855000; and had it been $\bar{2}$, the number would have been ·03855.

If the decimal part of the logarithm is not found exactly in the table, take the difference between the given logarithm and the next less tabular logarithm, for a numerator, and the difference between the next less and the next greater tabular logarithms, for a denominator. Reduce the fraction thus formed to a decimal, and, rejecting the decimal point, place the result at the right of the four figures corresponding to the less tabular logarithm; lastly, place a decimal point, if necessary, according to the characteristic of the given logarithm.

For example, let 2·4716423 be the logarithm, the corresponding number to which we wish to find. The next less decimal in the table is ·4715851, the difference between which and the given logarithm, the characteristic being neglected, is 572 of the lowest order of decimals in the logarithms; the difference between the next less and next greater tabular logarithms, is 1466 of the lowest order of decimals in the logarithms; reducing $\frac{572}{1466}$ to a decimal, we have the figures 39, which placed at the right of 2962, the figures corresponding to ·4715851, gives 296239; but as the characteristic of the given logarithm is 2, we point off three figures for integers, and obtain 296·239 for the required number.

The reason of this method is obvious. For, if, when the logarithm increases 1466, the number increases 1 unit of any order, when the logarithm increases 572, the number ought to increase $\frac{572}{1466}$ of 1 unit of the same order. The order of the unit of which we find a fractional part, is always determined by the characteristic. In the example just given we found the fractional part of 0·1, viz. ·039; but had the characteristic been 3, the fraction would have been a part of 1.

Let the learner find the numbers corresponding to the following logarithms, carrying the numbers to six significant figures, when the decimals are not found exactly in the tables.

Common tables will generally give numbers with sufficient accuracy to six or seven figures.

1.	1·4771213.	6.	0·2134445.
2.	3·3010300.	7.	$\bar{1}$·4840150.
3.	1·4991217.	8.	$\bar{2}$·7734667.
4.	3·1171167.	9.	$\bar{3}$·2276677.
5.	5·3458726.	10.	$\bar{4}$·3334475.

Art. **164**. Since the logarithm of a vulgar fraction is found by subtracting the logarithm of the denominator from that of the numerator, it follows that the logarithm of any proper fraction, like that of a decimal, must be negative. But we ordinarily make the characteristic only negative.

Thus, log. $\frac{2}{257}$ = log. 2 — log. 257 = 0·3010300 — 2·4099331 = $\bar{3}$·8910969. In order that we may be able to subtract the latter decimal from the former, we may suppose the characteristic 0 of the logarithm 0·3010300 to be changed into $-1+1$, so that the decimal ·4099331 can be taken from the positive part of $\bar{1} + 1$·3010300; then subtracting the 2 from -1, we have $\bar{3}$ for a characteristic. Or, as is more commonly done in subtraction, after having borrowed 1 in subtracting the left hand decimal, we may carry 1 to the 2, and then subtract the 3, which give $\bar{3}$, the same as before.

Art. **165**. But there is another form for the logarithm of any proper fraction, by which the negative characteristic is avoided. This form is obtained by increasing the true characteristic by 10.

For example, the logarithm of ·3 is $\bar{1}$·4771213; adding 10 to the characteristic and reducing gives log. ·3 = 9·4771213. In like manner, log. ·03 = 8·4771213; and log. ·003 = 7·4771213.

Hence, in this way, if the first figure of the decimal is tenths, the characteristic of its logarithm is 9; if the first figure is hundredths, the characteristic is 8; if the first figure is thousandths, the characteristic is 7, and so on. That is:

The characteristic of the logarithm of a decimal fraction, is 10 *diminished by as many units, as are equal to the number of places, by which the first significant figure of the fraction is removed from the decimal point.*

Likewise, in finding the logarithm of a vulgar fraction, we may increase the logarithm of the numerator by 10, and then subtract the logarithm of the denominator. Thus, the logarithm of $\frac{2}{257}$ would, in this way, be 7·8910969.

But we must recollect, that every such logarithm is, in fact, 10 too great, and that the result of any operation in which it is used, would be affected accordingly.

Art. **166**. In division, we have seen that the logarithm of the divisor is to be subtracted from that of the dividend; but, instead of this, we may add the *arithmetical complement* of the logarithm of the divisor to the logarithm of the dividend, dropping 10 afterwards in the result.

The arithmetical complement of the logarithm of a number, is what remains, after the logarithm of that number has been subtracted from 10

Thus, the arithmetical complement of the logarithm of 17, is $10 - \log. 17 = 10 - 1{\cdot}2304489 = 8{\cdot}7695511$.

The arithmetical complement of a logarithm may be found, by subtracting the first right hand significant figure of the logarithm from 10, and all the others from 9; so that we may, if we please, commence the subtraction at the left hand.

We must bear in mind that each arithmetical complement added, makes the result 10 too great, and allow for this in any operation, in which arithmetical complements of logarithms are used.

The fact that adding the arithmetical complement of a logarithm and afterwards subtracting 10, is equivalent to subtracting the logarithm itself, may be easily proved.

For, let l represent any logarithm, and l' a logarithm which is to be subtracted from it; the result would be $l - l'$. Now the arithmetical complement of l' is $10 - l'$; adding this to l,

we have $l+10-l'$; subtracting 10, we have $l+10-l'-10$, which reduced becomes $l-l'$, the same result as when l' was subtracted immediately from l.

If however we add the arithmetical complement of the logarithm of a fraction with its characteristic 10 too great, the result, without dropping 10, will be the same as if the logarithm of the fraction had been subtracted.

To prove this, let l' be the true logarithm of any fraction; then $10+l'$ would be its logarithm with a characteristic 10 too great; the arithmetical complement of this is $10-10-l'$, which added to any logarithm l, gives $l+10-10-l'$ or $l-l'$, which is precisely the same as if l' were directly subtracted from l.

Art. **167**. Let the learner find the logarithms of the following numbers. When the fractions are less than unity, the logarithms may be given in both forms, viz: with the negative characteristic, and with the characteristic 10 too great.

1. ·7234.	6. $\frac{378}{71}$.
2. ·00576.	7. $3\frac{7}{10}$.
3. ·00087926.	8. $456\frac{13}{17}$.
4. $\frac{2}{3}$.	9. $145\frac{1}{2}$.
5. $\frac{128}{775}$.	10. $\frac{3769}{17536}$.

Art. **168**. Find the numbers corresponding to the following logarithms, the characteristics being each 10 too great. Six significant figures may be found in each case, when the decimal part is not found exactly in the tables.

1. 9·4371213.	5. 5·4771213.
2. 8·7294179.	6. 9·8878879.
3. 6·3010300.	7. 8·8300379.
4. 2·6734217.	8. 7·4378678.

N. B. It will be observed that finding the logarithm of a vulgar fraction, and then obtaining the number corresponding to that logarithm, converts the fraction into a decimal.

SECTION LVIII.

APPLICATION OF LOGARITHMS TO ARITHMETICAL OPERATIONS.

Art. **169**. 1. Multiply 456 by 723.

$$\begin{array}{rl} \text{Log. } 456 = & 2{\cdot}6589648 \\ \text{Log. } 723 = & \underline{2{\cdot}8591383} \\ \text{Prod.} = 329688 \text{ - - - } & 5{\cdot}5181031. \end{array}$$

Adding the logarithms of 456 and 723, we find the sum to be 5·5181031; we then find from the tables the number corresponding to this logarithm, viz: 329688, which is the required product.

2. Multiply 2678 by ·03745.

$$\begin{array}{rl} \text{Log. } 2678 \;\; = & 3{\cdot}4278106 \\ \text{Log. } {\cdot}03745 = & \underline{\bar{2}{\cdot}5734518} \\ \text{Prod.} = 100{\cdot}29 \text{ - - - - - } & 2{\cdot}0012624. \end{array}$$

In adding, there is 1 to carry when we arrive at the characteristics; this 1 is positive, and, being added along with the 3 and $\bar{2}$, gives for a characteristic 4 — 2 or 2.

The same without the negative characteristic.

$$\begin{array}{rl} \text{Log. } 2678 \;\; = & 3{\cdot}4278106 \\ \text{Log. } {\cdot}03745 = & \underline{8{\cdot}5734518} \\ & 12{\cdot}0012624 \\ \text{Subtract - - - } & \underline{10\phantom{{\cdot}0012624}} \\ \text{Prod.} = 100{\cdot}29 \text{ - - - - - } & 2{\cdot}0012624. \end{array}$$

It would have saved labor to drop the 10 at the time of adding.

3. Multiply ·0374 by ·277.

$$\begin{array}{rl} \text{Log. } {\cdot}0374 = & \bar{2}{\cdot}5728716 \\ \text{Log. } {\cdot}277 \;\; = & \underline{\bar{1}{\cdot}4424798} \\ \text{Prod.} = {\cdot}0103598 \text{ - - - } & \bar{2}{\cdot}0153514. \end{array}$$

The figures answering to the decimal of the resulting logarithm are 103598; but since the characteristic is $\bar{2}$, the first

figure of the number must be hundredths, therefore a zero preceded by the decimal point, must be placed before the figures found.

The same without negative characteristics.

$$\begin{array}{rl} \text{Log. } {\cdot}0374 = & 8{\cdot}5728716 \\ \text{Log. } {\cdot}277 \;\; = & \underline{9{\cdot}4424798} \\ \text{Prod.} = {\cdot}0103598 \text{ - - -} & \overline{8}{\cdot}0153514. \end{array}$$

In the sum of the logarithms, the characteristic becomes, in fact, two tens too great; but we drop only one of them, and then the characteristic 8 shows that the first figure of the number must be hundredths.

4. Divide 48945 by 65.

$$\begin{array}{rl} \text{Log. } 48945 = & 4{\cdot}6897083 \\ \text{Log. } 65 \quad = & \underline{1{\cdot}8129134} \\ \text{Quot} = 753 \text{ - - - - - -} & 2{\cdot}8767949. \end{array} \Big\} \text{ By subtraction.}$$

The same with the arithmetical complement of the logarithm of the divisor. The contracted expression, *comp. log.*, will sometimes be used to signify arithmetical complement of the logarithm.

$$\begin{array}{rl} \text{Log. } \quad 48945 = & 4{\cdot}6897083 \\ \text{Comp. log. } 65 = & \underline{8{\cdot}1870866} \\ \text{Quot.} = 753 \text{ - - - - - - - -} & 2{\cdot}8767949. \end{array} \Big\} \text{ Add.}$$

5. Divide 775 by ·025.

$$\begin{array}{rl} \text{Log. } 775 = & 2{\cdot}8893017 \\ \text{Log. } {\cdot}025 = & \underline{\bar{2}{\cdot}3979400} \\ \text{Quot.} = 31000 \text{ - - -} & 4{\cdot}4913617. \end{array}$$

The sign of $\bar{2}$, in subtracting is changed to $+$, and then the two characteristics are added.

The same without the negative characteristic, and with the comp. log. of the divisor.

$$\begin{array}{rl} \text{Log. } 775 = \text{ - - - - - - - - - -} & 2{\cdot}8893017 \\ \text{Log. } {\cdot}025 = 8{\cdot}3979400, \text{ comp. log.} = & \underline{1{\cdot}6020600} \\ \text{Quot.} = 31000 \text{ - - - - - - - - - -} & 4{\cdot}4913617. \end{array}$$

In this example, the resulting characteristic is not too great, because the logarithm of the fraction was taken without the negative characteristic.

6. Divide ·005127 by ·0559.

$$\text{Log. } \cdot 005127 = \bar{3}\cdot 7098633$$
$$\text{Log. } \cdot 0559 \quad = \underline{\bar{2}\cdot 7474118}$$
$$\text{Quot.} = \cdot 091717 \text{ - - - - } \bar{2}\cdot 9624515.$$

In subtracting, there is 1 to carry to the $\bar{2}$, which makes it $\bar{1}$, and this subtracted from $\bar{3}$, gives $\bar{3}+1$ or $\bar{2}$ for a characteristic. Or, if the learner is accustomed, when he borrows 1, to diminish by 1 the next figure in the minuend, he will take 1 from $\bar{3}$, which gives $\bar{4}$; from this he will then subtract $\bar{2}$, and obtain $\bar{4}+2$, or $\bar{2}$.

The same with positive characteristics and comp. log. of the divisor.

$$\text{Log. } \cdot 005127 = \text{- - - - - - - - - - } 7\cdot 7098633$$
$$\text{Log. } \cdot 0559 = 8\cdot 7474118, \text{ comp. log.} = \underline{1\cdot 2525882}$$
$$\text{Quot.} = \cdot 091717 \text{ - - - - - - - - - } 8\cdot 9624515.$$

In this case the resulting logarithm is 10 too great, because the logarithm of the dividend was taken with a positive characteristic.

7. Required the third power of 27.

$$\text{Log. } 27 = 1\cdot 4313638$$
$$\underline{\qquad\qquad 3} \quad \text{Mult. by 3.}$$
$$\text{Power} = 19683 \text{ - - } 4\cdot 2940914.$$

8. Required the third power of ·271.

$$\text{Log. } \cdot 271 = \bar{1}\cdot 4329693$$
$$\underline{\qquad\qquad 3}$$
$$\text{Power} = \cdot 019902511 \text{ - - - - } \bar{2}\cdot 2989079.$$

The same with positive characteristics.

$$\text{Log. } {\cdot}271 = 9{\cdot}4329693$$
$$3$$
$$\text{Power} = {\cdot}019902511 \text{ - - } \overline{8{\cdot}2989079}.$$

By this last method, the characteristic after multiplication, becomes 28, which is three 10s or 30 too great; dropping two 10s or 20, we have the characteristic 8, which shows that the first figure of the number is hundredths.

9. Required the fifth root of 15.

$$\text{Log. } 15 = 1{\cdot}1760913(5 \quad \text{Divide by } 5.$$
$$\text{Root} = 1{\cdot}71877 \text{ - - - } \overline{0{\cdot}2352183}.$$

10. Find the third root of ·000729.

$$\text{Log. } {\cdot}000729 = \bar{4}{\cdot}8627275 = \bar{6} + 2{\cdot}8627275(3$$
$$\text{Root} = {\cdot}09 \text{ - - - - - - - - - - } \bar{2}{\cdot}9542425.$$

In this question a difficulty occurs in dividing the logarithm $\bar{4}{\cdot}8627275$, since the integral and fractional parts have different signs, and the negative characteristic is not divisible by 3. To obviate this difficulty, add $\bar{2}+2$, which is zero, to the characteristic; the logarithm then becomes $\bar{6}+2{\cdot}8627275$. Dividing now the negative and positive parts separately, we have the result as above.

In all cases of finding the root of a fraction, if its logarithm is taken with a negative exponent, and that exponent is not divisible by the number expressing the degree of the root, we must make it so, by adding to the logarithm $\bar{1}+1$, $\bar{2}+2$, $\bar{3}+3$, or some equivalent expression.

The same with positive characteristics.

$$\text{Log. } {\cdot}000729 = 6{\cdot}8627275$$
$$20 \text{ - - - - Add.}$$
$$\overline{26{\cdot}8627275}(3$$
$$\text{Root} = {\cdot}09 \text{ - - - - } \overline{8{\cdot}9542425}.$$

By the second method, the logarithm when first found, is too great by 10; we then add two more 10s, which makes it three 10s too great; this divided by 3 gives a result 10 too great as required.

Whenever we use the positive characteristic in finding the root of a fraction, before dividing the logarithm, it is necessary to make the characteristic as many 10s too great as there are units in the number which marks the degree of the root. The division will then leave the result one 10 too great.

11. Find the value of x in the expression, $x = (\frac{2}{7})^{\frac{3}{5}}$.

Log. 2 = - - - - - - - - - - - -	0·3010300
Comp. log. 7 = - - - - - - - - - -	9·1549020
Log. $\frac{2}{7}$ - - char. 10 too great, = - - -	9·4559320
	3
Log. $(\frac{2}{7})^3$ - - char. three 10s too great, =	28·3677960
	20 - - - - - Add.
	48·3677960(5
$x =$ ·471584 - - - - - - - - - - - -	9·6735592.

12. Find the value of x in the expression, $x = \left(\frac{45\ .\ 13\ .\ (\cdot 75)}{19\ .\ 117\ .\ 11}\right)^{\frac{3}{4}}$.

Log. 45 = - - - - -	1·6532125
Log. 13 = - - - - -	1·1139434
Log. ·75 = - - - - -	9·8750613
Comp. log. 19 =	8·7212464
Comp. log. 117 =	7·9318141
Comp. log. 11 =	8·9586073
	8·2538850
	3
	24·7616550
	10
	34·7616550(4
$x =$ ·0490245 - - - -	8·6904137.

In this question we have used the logarithm of one fraction, with the increased characteristic, and three comp. logs. of whole numbers; the sum of the six logarithms added, will therefore be 40 too great. Dropping 30, multiplying by 3, adding 10 to the product, and dividing this sum by 4, will leave the final logarithm 10 too great.

13. Find the value of x in the expression,

$$x = \frac{\sqrt{\tfrac{2}{3}} \cdot \sqrt[3]{\tfrac{7}{8}} \cdot (\cdot 075)}{(12)^3 \cdot (\tfrac{3}{5})^4 \cdot \sqrt[5]{\tfrac{4}{5}}}.$$

Log. 2	= 0·3010300		
Comp. log. 3	= 9·5228787		
	9·8239087		
	10·		
	19·8239087 (2		
Log. $\sqrt{\tfrac{2}{3}}$	= 9·9119543	- - - - - - - - - -	9·9119543.
Log. 7	= 0·8450980		
Comp. log. 8	= 9·0969100		
	9·9420080		
	20·		
	29·9420080 (3		
Log. $\sqrt[3]{\tfrac{7}{8}}$	= 9·9806693	- - - - - - - - - -	9·9806693.
Log. ·075	= - - - - - - - - - - - -		8·8750613.
Log. 12	= 1·0791812		
	3		
Log. $(12)^3$	= 3·2375436	- - comp. log.	= 6·7624564.
Log. 3	= 0·4771213		
Comp. log. 5	= 9·3010300		
Log. $\tfrac{3}{5}$	= 9·7781513		
	4		
Log. $(\tfrac{3}{5})^4$	= 9·1126052	- - comp. log.	= 0·8873948.

$$\begin{array}{ll}
\text{Log. } 4 & = 0{\cdot}6020600 \\
\text{Comp. log. } 9 & = \underline{9{\cdot}0457575} \\
\text{Log. } \frac{4}{9} & = 9{\cdot}6478175 \\
 & \quad \underline{40{\cdot}} \\
 & \quad \underline{49{\cdot}6478175}\,(5 \\
\text{Log. } \sqrt[5]{\frac{4}{9}} & = 9{\cdot}9295635 \text{ - - - comp. log. } = 0{\cdot}0704365.
\end{array}$$

We now add the several results which are carried out to the right.

$$\begin{array}{ll}
\text{Log. } \sqrt{\frac{2}{3}} & = 9{\cdot}9119543 \\
\text{Log. } \sqrt[3]{\frac{7}{8}} & = 9{\cdot}9806693 \\
\text{Log. } {\cdot}075 & = 8{\cdot}8750613 \\
\text{Comp. log. } (12)^3 & = 6{\cdot}7624564 \\
\text{Comp. log. } (\frac{3}{5})^4 & = 0{\cdot}8873948 \\
\text{Comp. log. } \sqrt[5]{\frac{4}{9}} & = \underline{0{\cdot}0704365} \\
x = {\cdot}00030695 \text{ - - - - - - } & \;\; 6{\cdot}4870726.
\end{array}$$

Some labor might have been saved in this problem, by substituting equivalents for several of the quantities, viz: ·875 for $\frac{7}{8}$, 1728 for $(12)^3$, and ·6 for $\frac{3}{5}$. But the object was, to exhibit the general mode of proceeding, and not the shortest for this particular case.

Although in several of the preceding problems, logarithms of fractions have been used in both forms, it is advisable, in most cases, to use the increased characteristic; especially as the learner who is to study Trigonometry, will have occasion to use tables in which every characteristic is 10 too great.

Perform the following questions by means of logarithms.

14. Multiply 37·153 by 4·086.
15. Multiply 257·3 by 300.
16. Multiply 567 by ·572.
17. Multiply ·0387 by ·093.
18. Multiply $\frac{2}{3}$ by 11·5756.
19. Multiply $\frac{275}{856}$ by $9\frac{3}{8}$.

20. Multiply $147\frac{3}{5}$ by $24\frac{5}{19}$.
21. Find the product of 375, 325, and ·03756.
22. Divide 12783 by 256.
23. Divide 147324 by 24·8333.
24. Divide 225·63 by ·0473.
25. Divide ·0743 by ·3967.
26. Divide $\frac{4}{17}$ by $\frac{3}{119}$.
27. Divide $126\frac{3}{8}$ by $17\frac{3}{4}$.
28. Find the 4th power of 2·73.
29. Find the 3d power of 9·16.
30. Find the 5th power of ·03.
31. Find the 5th power of $2\frac{7}{9}$.
32. Find the 2d root of 5.
33. Find the 3d root of 42·3.
34. Find the 3d root of ·0756.
35. Find the 4th root of ·37.
36. Find the 7th root of ·951.
37. Find the 5th root of $\frac{27}{55}$.
38. Find the value of $(\frac{3}{7})^{\frac{2}{5}}$.
39. Find the value of $(\frac{233}{491})^{\frac{2}{7}}$.
40. Find the value of $\sqrt{49 \,.\, \frac{3}{8} \,.\, (\cdot 0673)}$.
41. Find the value of $(\frac{25}{49})^{\frac{2}{3}} \,.\, (\frac{107}{119})^{\frac{3}{5}}$.
42. Find the value of $\sqrt[5]{\frac{25}{4}} \,.\, \sqrt{(\frac{3}{8})^2}$.
43. Find the value of $\dfrac{\sqrt{3} \,.\, (\cdot 073) \,.\, 256}{\sqrt[3]{\frac{2}{5}} \,.\, (\frac{6}{11})^{\frac{2}{3}} \,.\, (\cdot 056)^{\frac{1}{3}}}$.
44. Find the value of x in the equation, $55^x = 493$.

Such an expression as 55^x, in which the exponent is unknown, is called an *exponential quantity*.

Since the logarithm of any power of a quantity, is found by multiplying the logarithm of that quantity by the number which expresses the degree of the power, we have, in the present case, by taking the logarithms of both members,

$x \times \log. 55 = \log. 493$, or,
$x \times 1{\cdot}7403627 = 2{\cdot}6928469$. Hence,
$x = \frac{2{\cdot}6928469}{1{\cdot}7403627} = 1{\cdot}5473$.

The division, performed in the common way, gives $x = 1{\cdot}5473$.

But we may take the logarithms of these logarithms, as we would of any other numbers, and perform the division as usual with logarithms.

Log. 2·6928469	=	0·4302117
Comp. log. 1·7403627	=	9·7593603
$x = 1{\cdot}5473$ - - - - - - - - -		0·1895720.

Let the learner perform the following questions, finding five figures in the answer to each.

45. Find x in $4^x = 27$.
46. Find x in $7^x = 9$.
47. Find x in $12^{\frac{3}{x}} = 44$.

In the last question, raise both members to the xth power, which gives $44^x = 12^3$, or $44^x = 1728$; the value of x may then be found as in the preceding examples.

48. Find x in the proportion, $720 : 196 = 155{\cdot}5 : x$.

We know from the principles of proportion that $x = \frac{196 \,.\, 155{\cdot}5}{720}$; hence, we are to add together the logarithms of the means, and the comp. log. of the first term. We may therefore begin with the first term.

Comp. log. 720	=	7·1426675
Log. 196	=	2·2922561
Log. 155·5	=	2·1917304
$x = 42{\cdot}33$ - - - - - -		$\overline{1}{\cdot}6266540$.

For the convenience of applying logarithms, the terms of a proportion may be placed under each other, care being taken to change the order of the terms, if necessary, so that the unknown shall stand last, and to use the comp. log. of the first term in that arrangement.

49. Find x in the proportion, $15 : x = 100 : 47$.

50. Find a mean proportional between 12·5 and 75·83.

51. Insert four mean proportionals between 7 and 20.

52. Required the sum of a progression by quotient, the first term being 5, the ratio 4, and the number of terms 6.

Substituting the given numbers in the formula, $\dfrac{a\,(q^n - 1)}{q-1}$, we have $S = \dfrac{5\,(4^6 - 1)}{3}$.

$$
\begin{array}{rll}
 & \text{Log. } 4 & = 0\cdot6020600 \\
 & & \quad\quad\quad\;\; 6 \\
4^6 = 4096 & \text{--------} & \overline{3\cdot6123600} \\
 1 & & \\
4^6 - 1 = \overline{4095}, & \text{its log.} & = 3\cdot6122539 \\
 & \text{Log. } 5 & = 0\cdot6989700 \\
 & \text{Comp. log. } 3 & = 9\cdot5228787 \\
S = 6825 & \text{--------} & \overline{3\cdot8341026}.
\end{array}
$$

Art. **170**. We may now solve the four questions in progression by quotient mentioned in Art. **151**, assuming the formulæ, $l = a\,q^{n-1}$, and $S = \dfrac{q\,l - a}{q-1}$. The solution of one of them will be given, and that of the others will be left as an exercise to the learner.

1. Given a, q and l; to find S and n.

The value of S is already given, viz: $S = \dfrac{q\,l - a}{q-1}$.

To find n; the equation, $l = a\,q^{n-1}$, gives, in succession,

$$q^{n-1} = \frac{l}{a},$$

$$(n-1)\,\text{log. } q = \text{log.}\left(\frac{l}{a}\right) = \text{log. } l - \text{log. } a; \text{ hence,}$$

$$n - 1 = \frac{\text{log. } l - \text{log. } a}{\text{log. } q}, \text{ and}$$

$$n = \frac{\text{log. } l - \text{log. } a}{\text{log. } q} + 1.$$

2. Given a, l and S; find q and n.
3. Given a, q and S; find l and n.
4. Given q, l and S; find a and n.

The following questions may be solved by means of the formulæ obtained from the four preceding problems.

5. The first term of a progression by quotient being 3, the ratio 2, and the last term 6144; required the sum and the number of terms.

6. The first term of a progression by quotient is 6, the last term 13122, and the sum 19680; required the ratio and the number of terms.

7. The first term of a progression by quotient being 9, the ratio 3, and the sum 265716; required the number of terms and the last term.

8. The sum of a progression by quotient being 6560, the ratio 3, and the last term 4374; required the first term and the number of terms.

SECTION LIX.

COMPOUND INTEREST.

Art. **171**. Let p represent any sum of money put at compound interest, for a number t of years, at the annual rate of r per cent., r being a decimal, as $\cdot 05$ or $\cdot 06$. It is required to find the amount, which we represent by A.

It is manifest, that, if any principal be multiplied by $1 +$ the rate, the product will be the amount for one year; for this is the same as multiplying the principal by the rate, which gives the interest for one year, and adding the result to the principal. Thus, the amount of \$10, for a year at 6 per cent., is $10\,(1\cdot 06)$ or \$10·60.

The amount, then, of p dollars for one year, is $p\,(1+r)$; this is the capital for the second year, and, to obtain the amount at

the end of that year, we must multiply this capital by $1+r$, which gives $p(1+r)^2$; this being the capital for the third year, and being multiplied by $1+r$, gives, for the amount at the end of the third year, $p(1+r)^3$. In like manner, the amount at the end of the fourth year is $p(1+r)^4$; that at the end of the fifth year is $p(1+r)^5$.

The amount in any case, therefore, is found by raising 1 + the rate to the power denoted by the number of years, and multiplying the result by the principal.

The formula for the amount, therefore, is

$$A = p(1+r)^t.$$

1. Required the amount of \$750, for 4 years at 6 per cent., compound interest.

In this question, $p = 750$, $r = \cdot 06$, and $t = 4$. Substituting these numbers in the formula, we have $A = 750\,(1\cdot 06)^4$.

$$\begin{array}{rl}
\text{Log. } 1\cdot 06 & = 0\cdot 0253059 \\
 & \quad\quad\quad\quad\;\; 4 \\
\text{Log. } (1\cdot 06)^4 & = \overline{0\cdot 1012236} \\
\text{Log. } 750 & = 2\cdot 8750613 \\
A = \$946\cdot 858 \text{ - - - - - -} & \overline{2\cdot 9762849}.
\end{array}$$

2. Required the amount of \$1050, for $5\frac{1}{2}$ years at 5 per cent., compound interest.

$$\begin{array}{rl}
\text{Log. } 1\cdot 05 & = 0\cdot 0211893 \\
 & \quad\quad\quad\quad\;\; 5\tfrac{1}{2} \\
 & \;\;\; \overline{0\cdot 1059465} \\
 & \;\;\; 0105946 \\
\text{Log. } (1\cdot 05)^{5\frac{1}{2}} & = \overline{0\cdot 1165411} \\
\text{Log. } 1050 & = 3\cdot 0211893 \\
A = \$1373\cdot 19 \text{ - - - - - -} & \overline{3\cdot 1377304}.
\end{array}$$

It is common with merchants, to find the amount for the number of whole years, and then find, at simple interest, the amount of that sum for the fractional part of a year. According to this method, the process by logarithms would be as follows.

Log. 1·05 = 0·0211893
5
Log. $(1·05)^5$ = 0·1059465
Log. 1050 = 3·0211893
Log. of am. for 5 years = 3·1271358
Log. (1·025) = 0·0107239
$A =$ \$1373·598 - - - - - - - 3·1378597.

After having found the logarithm of the amount for 5 years, we add to it the logarithm of 1·025, that is, of 1 + the rate for six months.

The last result exceeds that obtained in the previous solution by \$0·408. In succeeding questions the former method may be pursued.

Any three of the four quantities in the equation, $A = p\,(1+r)^t$, being known, the remaining one may be found. Making p, r and t successively the unknown quantity, we obtain the following formulæ.

$$p = \frac{A}{(1+r)^t}.$$

$$r = \left(\frac{A}{p}\right)^{\frac{1}{t}} - 1.$$

$$t = \frac{\log.\left(\dfrac{A}{p}\right)}{\log.\,(1+r)}.$$

3. What sum must be put at interest, the rate being 6 per cent., in order to amount to \$1287 in 4 years?

In this question p is to be found, and the formula, $p = \frac{A}{(1+r)^t}$, by the substitution of the given quantities, becomes

$$p = \frac{1287}{(1·06)^4}.$$

Log. 1·06 = 0·0253059
4
Log. $(1·06)^4$ = 0·1012236 - - - comp. log. = 9·8987764
Log. 1287 = - - - - - - - - - - - - - 3·1095785
$p =$ \$1019·424 - - - - - - - - - - - - 3·0083549.

The value of p in this example is called the present worth of A.

4. The principal \$400 amounts, in 9 years, at compound interest, to \$569·333; required the rate per cent.

Substituting in the formula, $r = \left(\frac{A}{p}\right)^{\frac{1}{t}} - 1$, we have $r = \left(\frac{569 \cdot 333}{400}\right)^{\frac{1}{9}} - 1$.

$$\begin{array}{rl} \text{Log. } 569 \cdot 333 & = 2 \cdot 7553664 \\ \text{Comp. log. } 400 & = \underline{7 \cdot 3979400} \\ & \overline{0 \cdot 1533064}\,(9 \\ 1 + r = 1 \cdot 04 \text{ - - - - -} & 0 \cdot 0170340. \\ r = \cdot 04. & \end{array}$$

5. How many years must \$1000 remain at compound interest, the rate being 6 per cent., in order to amount to \$1191·016?

Substituting in the formula, $t = \frac{\log. \left(\frac{A}{p}\right)}{\log. (1 + r)}$, we obtain $t = \frac{\log. \left(\frac{1191 \cdot 016}{1000}\right)}{\log. (1 \cdot 06)}$.

$$\begin{array}{ll} \text{Log. } \ 1191 \cdot 016 & = 3 \cdot 0759176 \\ \text{Comp. log. } 1000 & = \underline{7 \cdot 0000000} \\ \text{Log. } \left(\frac{A}{p}\right) & = 0 \cdot 0759176 \\ \text{Log. } 1 \cdot 06 & = 0 \cdot 0253059. \end{array}$$

Hence, $t = \frac{\cdot 0759176}{\cdot 0253059} = 3$ years.

Or, performing this last division by logarithms, we have

$$\begin{array}{ll} \text{Log. } \cdot 0759176 & = 8 \cdot 8803424 \\ \text{Comp. log. } \cdot 0253059 & = \underline{1 \cdot 5967783} \\ t = 3 \text{ - - - - - - -} & 0 \cdot 4771207. \end{array}$$

In this question the operation would have been shorter, if we had divided 1191·016 by 1000, before applying logarithms.

6. Find the amount, at compound interest, of \$357·50, for 8 years at 6 per cent.

7. Find the amount of \$1573 for 4 years, at $5\frac{1}{2}$ per cent. compound interest.

8. Required the compound interest on \$1000, for 7 years and 4 months at 4 per cent.

9. What sum of money will, in 6 years, at 7 per cent. compound interest, amount to \$2745·90?

10. What sum of money will amount, in 10 years, to \$447·712, compound interest being reckoned at 6 per cent.?

11. In how many years will \$75 amount to \$149·495, at 5 per cent. compound interest?

12. A principal of \$108·50 amounted, in 12 years, at compound interest, to \$220·45; what was the rate per cent.?

13. In what time would any sum be doubled at compound interest, the rate being 6 per cent.?

In this question the amount is to become double the principal; therefore, in the formula for t, we substitute $2p$ instead of A, which gives $t = \frac{\log.\left(\frac{2p}{p}\right)}{\log.(1+r)}$, or, by reduction, $t = \frac{\log. 2}{\log.(1+r)}$.

14. In how many years will any sum, at compound interest, be tripled, the rate being 6 per cent.?

15. In how many years will any sum be doubled, at 5 per cent. compound interest?

16. What would \$357 amount to in 10 years at compound interest, the interest being reckoned semi-annually, at the rate of 6 per cent. a year?

17. The population of Boston in 1830 was 61392; what was it in 1840, supposing the annual rate of increase to be $3\frac{1}{10}$ per cent.?

18. The population of Philadelphia in 1830 was 188797, and in 1840 it was 258832: what was the annual rate of increase?

19. In 1830 New York contained 202589, and in 1840 it contained 312234 inhabitants; if the population continue to increase at the same rate as it did from 1830 to 1840, in how many years from the latter date will it amount to 1000000?

Art. **172.** 1. A man saves annually \$300 which, at the end of each year, he deposites in a bank, and is allowed 5 per cent. compound interest. How much would be due him from the bank, at the end of 12 years from the time of the first deposit?

To generalize this question, let a be the sum annually deposited, t the time, and r the rate. Then the amount of the sum first deposited would, according to the principles already given, be $a(1+r)^t$. The second deposit remaining in the bank one year less, would amount to $a(1+r)^{t-1}$. The amount of the third deposit would be $a(1+r)^{t-2}$, and so on. The last deposit but one, remaining in the bank two years, would amount to $a(1+r)^2$; and the last deposit would amount to $a(1+r)$.

Hence, if A represent the gross amount, we have

$$A = a(1+r) + a(1+r)^2 + \ldots\ldots + a(1+r)^{t-2} + a(1+r)^{t-1} + a(1+r)^t.$$

The second member of this equation is a progression by quotient, in which the first term is $a(1+r)$, the ratio $1+r$, and the last term $a(1+r)^t$. In the formula, $S = \frac{lq-a}{q-1}$, substituting A instead of S, $a(1+r)^t$ instead of l, $a(1+r)$ instead of a, and $1+r$ instead of q, we have

$$A = \frac{a(1+r)^t(1+r) - a(1+r)}{1+r-1}; \text{ or}$$

$$A = \frac{a(1+r)[(1+r)^t - 1]}{r}.$$

Substituting in this formula the numbers given in the question proposed, we have

$$A = \frac{300(1{\cdot}05)[(1{\cdot}05)^{12} - 1]}{{\cdot}05}.$$

In applying logarithms, it is best to commence with the quantity between the brackets.

$$\begin{array}{rlr}
 & \text{Log. } 1{\cdot}05 = & 0{\cdot}0211893 \\
 & & 12 \\
\hline
(1{\cdot}05)^{12} = & 1{\cdot}795856 \text{ - - } & 0{\cdot}2542716 \\
\hline
 & 1 & \\
\hline
\text{Log.} & {\cdot}795856 = & 9{\cdot}9008345 \\
\text{Log.} & 1{\cdot}05 \quad = & 0{\cdot}0211893 \\
\text{Log.} & 300 \quad = & 2{\cdot}4771213 \\
\text{Comp. log.} & {\cdot}05 = & 1{\cdot}3010300 \\
\hline
A = \$5013{\cdot}893 & \text{- - - -} & 3{\cdot}7001751.
\end{array}$$

2. If a young man, by omitting some useless expense, saves 25 cents every day, and, at the end of each year, deposits his savings in an institution which allows 6 per cent. compound interest, how much would be due him from the institution, at the end of 20 years from the time of the first deposit, a year being considered 365 days?

SECTION LX.

ANNUITIES.

Art. **173**. An annuity is a certain sum of money payable annually, or at other regular periods, for a stated number of years, or during a person's life, or forever. The following question is one of annuities.

1. A man wishes to put at compound interest such a sum of money, as will afford him annually $500 for 20 years, at the end of which time the principal and interest shall be exhausted. What sum must he put at interest, the rate being 6 per cent.?

It is manifest that the amount of all he receives, must be the same as the amount of the sum put at interest.

To generalize this question, let a be the sum received annually, r the rate of interest, and t the time.

As the first sum is drawn out at the end of the first year, the drawer must be considered as having received, at the expiration of the whole time, the amount of that sum at compound interest for $t-1$ years, which according to Art. **171**, is $a(1+r)^{t-1}$. In like manner, that drawn out at the end of the second year, amounts to $a(1+r)^{t-2}$; that at the end of the third year, to $a(1+r)^{t-3}$, and so on; the sum drawn at the end of the last year is simply a.

The gross amount of the whole drawn out, is, therefore,

$a(1+r)^{t-1}+a(1+r)^{t-2}+a(1+r)^{t-3}+\ldots\ldots+a(1+r)^2+a(1+r)+a$; or, by a change in the order of arrangement,

$a+a(1+r)+a(1+r)^2+\ldots\ldots+a(1+r)^{t-3}+a(1+r)^{t-2}+a(1+r)^{t-1}$.

This is a progression by quotient, in which the first term is a, the ratio $1+r$, and the last term $a(1+r)^{t-1}$. Substituting these in the formula, $S=\dfrac{q\,l-a}{q-1}$, we have

$$S=\frac{a(1+r)^{t-1}(1+r)-a}{r}=\frac{a(1+r)^t-a}{r}=\frac{a[(1+r)^t-1]}{r}.$$

Now let A be the sum put at interest. This would amount in t years to $A(1+r)^t$; and since this amount must be equal to that of the several sums drawn out, we have

$$A(1+r)^t=\frac{a[(1+r)^t-1]}{r};\ \text{hence,}$$

$$A=\frac{a[(1+r)^t-1]}{r(1+r)^t}.$$

Substituting the numbers given in the proposed question, we have $A=\dfrac{500[(1{\cdot}06)^{20}-1]}{{\cdot}06\,(1{\cdot}06)^{20}}$.

$$
\begin{array}{lrl}
\text{Log. } 1{\cdot}06 & = & 0{\cdot}0253059 \\
 & & \underline{\quad\quad\quad 20} \\
(1{\cdot}06)^{20} = 3{\cdot}20714 \text{ - -} & & \underline{0{\cdot}5061180} \\
\quad\quad\quad\quad \underline{1} & & \\
\text{Log. } 2.20714 & = & 0{\cdot}3438299 \\
\text{Log. } 500 & = & 2{\cdot}6989700 \\
\text{Comp. log. } {\cdot}06 & = & 1{\cdot}2218487 \\
\text{Comp. log. } (1{\cdot}06)^{20} & = & \underline{9{\cdot}4938820} \\
A = \$5734{\cdot}963 \text{ - - - -} & & 3{\cdot}7585306.
\end{array}
$$

In the equation, $A = \dfrac{a\,[(1+r)^t - 1]}{r(1+r)^t}$, we may make either of the quantities, A, a, t and r, the unknown. Thus, to find a, we have successively, $a\,[(1+r)^t - 1] = A\,r\,(1+r)^t$;

$$a = \frac{A\,r\,(1+r)^t}{(1+r)^t - 1}.$$

To find t we obtain successively from the equation, $A = \dfrac{a\,[(1+r)^t - 1]}{r\,(1+r)^t}$,

$$a\,(1+r)^t - a = A\,r\,(1+r)^t;$$

$$a\,(1+r)^t - A\,r\,(1+r)^t = a;$$

$$(a - A\,r)\,(1+r)^t = a;$$

$$(1+r)^t = \frac{a}{a - A\,r};$$

$$t \times \log.\,(1+r) = \log.\left(\frac{a}{a - A\,r}\right);$$

$$t = \frac{\log.\left(\dfrac{a}{a - A\,r}\right)}{\log.\,(1+r)}.$$

To find r would be too difficult for the design of this treatise.

2. If a person deposite \$5000 in an annuity office, how much can he draw annually, if the annuity is to continue 10 years, compound interest being reckoned at 5 per cent.?

In this question, a is the unknown quantity, and the formula for a, by substitution, gives

$$a = \frac{5000 \,.\, (\cdot 05)\,(1\cdot 05)^{10}}{(1\cdot 05)^{10} - 1} = \frac{250\,(1\cdot 05)^{10}}{(1\cdot 05)^{10} - 1}.$$

In applying logarithms, it is best, in this case, to commence with the denominator.

$$\begin{array}{rll}
\text{Log. } 1\cdot 05 & = & 0\cdot 0211893 \\
 & & \underline{10} \\
(1\cdot 05)^{10} = 1\cdot 62889 & \text{- -} & \overline{0\cdot 2118930} \\
\underline{1} & & \\
\text{Comp. log. } \cdot 62889 & = & 0\cdot 2014253 \\
\text{Log. } 250 & = & 2\cdot 3979400 \\
\text{Log. } (1\cdot 05)^{10} & = & \underline{0\cdot 2118930} \\
a = \$647\cdot 527 & \text{- - - -} & 2\cdot 8112583.
\end{array}$$

3. A man deposits in an annuity office \$7500, for which $5\frac{1}{2}$ per cent. compound interest is allowed; in how many years will it be exhausted, if he draws out annually \$750?

The formula for t, by substitution, gives $t =$

$$\frac{\log.\left(\dfrac{750}{750 - (\cdot 055)\,.\,7500}\right)}{\log.\,(1\cdot 055)}, \text{ or, by reduction, } t = \frac{\log.\left(\dfrac{750}{337\cdot 50}\right)}{\log.\,(1\cdot 055)}.$$

$$\begin{array}{rll}
\text{Log. } 750 & = & 2\cdot 8750613 \\
\text{Comp. log. } 337\cdot 50 & = & 7\cdot 4717262 \\
\text{Log. } (\frac{750}{337\cdot 50}) & = & \overline{0\cdot 3467875} \\
\text{Log. } 1\cdot 055 & = & 0\cdot 0232525. \quad \text{Hence,} \\
t = \frac{\cdot 3467875}{\cdot 0232525}. & & \\
\text{Log. } \cdot 3467875 & = & 9\cdot 5400634 \\
\text{Comp. log. } \cdot 0232525 & = & 1\cdot 6335303 \\
t = 14\cdot 914 \text{ years} & \text{- - - -} & \overline{1\cdot 1735937}; \\
\end{array}$$

or, $t = 14$ years, 10 months, and 29 days.

4. A gentleman wishes to purchase an annuity, which shall afford him \$500 annually for 30 years; how much must he pay if he is allowed 5 per cent. interest?

5. How much must be given for an annuity to last 20 years, if $300 are to be drawn semi-annually, and interest be allowed at the rate of $4\frac{1}{2}$ per cent. a year?

6. A gentleman purchases an annuity for the benefit of his family after his decease, and pays $10000. Three years from the date of the purchase he dies, and then the annuity comes into operation. How much must the family draw out annually, so as to exhaust the annuity in 15 years from the time it commences, if $5\frac{1}{2}$ per cent. interest be allowed?

In this question A must be the amount of $10000 for 3 years.

7. How long would the annuity in the last question continue, on condition that the family received $1000 annually?

MISCELLANEOUS QUESTIONS.

1. Four men, A, B, C and D, bought a ship for $10428; of which B paid twice as much as A, C paid as much as A and B, and D paid as much as B and C. How much did each pay?

2. A person bought 8 yards of cloth for £3 2s, giving 9s a yard for a part, and 7s a yard for the rest. How many yards did he buy at each price?

3. A father is 40 years old, and his son 8; in how many years will the father be three times as old as the son?

4. A young man spends $\frac{1}{4}$ of his annual income for board, and $\frac{1}{2}$ as much for clothes; his other incidental expenses amount to $\frac{1}{2}$ as much as his clothes, and yet he saves $490 a year. What is his yearly income?

5. A person had spent $\frac{1}{4}$ of his life in England, $\frac{1}{5}$ of it on the continent of Europe, 5 years more than $\frac{1}{12}$ of it in Asia, and 3 years more than $\frac{1}{3}$ of it in America. How old was he?

6. What number is that, from which if 5 be subtracted, and the remainder be divided by 2, and again if 5 be subtracted from this quotient, and the remainder be divided by 2, it will leave $\frac{2}{11}$ of the number itself?

7. A man could reap a field of wheat in 5 days, and his son could reap it in 20 days. In what time would they together reap it?

8. If a certain number be subtracted from 100 and 120 respectively, $\frac{1}{4}$ of the former remainder will be equal to $\frac{1}{5}$ of the latter. Required the number.

9. There is a rectangular piece of land, whose length exceeds its breadth by 10 rods; if the field were a square whose side was equal to its present length, it would contain 400 square rods more than it now contains. Required the length and breadth.

10. To pay a debt of £39 with 40 coins, eagles and dollars, how many of each must I have, the dollar being 6 shillings?

11. Two men, A and B, had together $108; the former spent $\frac{1}{3}$ and the latter $\frac{1}{4}$ of what he had; and the amount of what both spent was $32. How much money had each at first?

12. A merchant commencing business with a certain capital, lost $\frac{1}{2}$ of it the first year; but the next year he gained $700; he thus continued alternately losing $\frac{1}{2}$ of what he had at the time, and gaining $700, until, at the end of the 6th year, he had $350 more than he commenced with at first. With what capital did he commence?

13. A, B and C had the same amount of money; A gave away $5, and spent $\frac{1}{3}$ of the remainder; B gave away $10, and spent $\frac{1}{5}$ of the remainder; C gained $10, and spent $\frac{1}{10}$ of what he then had; after which they had together $116. How much money had each at first?

14. A father leaves to his three sons £1600, in the following manner. The second is to have £200 less than the eldest, and £100 more than the youngest. Required the share of each.

15. Of a battalion of men, $\frac{3}{4}$ of the whole are on duty, $\frac{1}{10}$ are sick, $\frac{3}{5}$ of the remainder are absent, and there are 48 officers. How many persons are there in the battalion?

16. A and B found a purse containing dollars. A took from

it \$2, and $\frac{1}{6}$ of the remainder; after which B took from it \$3, and $\frac{1}{6}$ of the remainder, when it was found that A and B had taken out equal sums. How much money was there in the purse at first?

17. A and B have the same yearly income; A contracts an annual debt amounting to $\frac{1}{7}$ of his; while B spends only $\frac{4}{5}$ of his. At the end of 10 years B lends A money enough to pay the debt which he has contracted in the mean time, and has £160 left. What is the income of each?

18. A gentleman found, that, in order to give some beggars 2s 6d each, he would want 3s; he therefore gave 2s to each, and had 4s left. How many beggars were there, and how much money had the gentleman?

19. Find a number, such, that whether it be divided into two or three equal parts, the continued product of the parts shall be of the same value.

20. Divide 72 into three parts, so that $\frac{1}{2}$ of the first shall be equal to the second, and $\frac{3}{5}$ of the second shall be equal to the third.

21. A man bought 6 bushels of wheat and 3 bushels of rye for \$13; he afterwards sold 4 bushels of wheat and 7 bushels of rye at the same rate for \$13$\frac{2}{3}$. How many shillings were given a bushel for each?

22. There is a certain fraction, to the numerator which, if 3 be added, the value of the fraction will be $\frac{1}{3}$; but if 1 be subtracted from the denominator, the value of the fraction will be $\frac{1}{5}$. What is the fraction?

23. There is a number consisting of two digits. The sum of the digits is 5; and if 9 be added to the number itself, the digits will be inverted. Required the number.

24. The sum of two numbers is 37; and if three times the less be subtracted from four times the greater, $\frac{1}{6}$ of the difference will be 6. Required the numbers.

25. Separate 25 into two parts, such that their product shall be 136.

26. A gambler lost $\frac{1}{4}$ of his money, and then won 3 shillings; again he lost $\frac{1}{3}$ of what he then had, and afterwards won 2 shillings; lastly, he lost $\frac{1}{8}$ of what he then had, and found that he had 14 shillings left. How much money had he at first?

27. There is a number consisting of two digits, to the sum of which, if 7 be added, the result will be equal to three times the left hand digit; but if 18 be subtracted from the number itself, the digits will be inverted. Required the number.

28. Says A to B, give me \$15 of your money, and I shall have as much as you will have left; true, says B, but give me \$10 of your money, and I shall have six times as much as you will have left. How much money has each?

29. A vintner has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11 gallons, and the quantities remaining are as 8 to 3. After the casks are half emptied, he puts 10 gallons of water into each, and the quantities of liquor then in them are as 9 to 5. How much does each cask hold?

30. A and B speculate with different sums of money; A gains £150, and B loses £50; then A's stock is to B's as 3 to 2. But had A lost £50, and B gained £100, A's stock would have been to B's as 5 to 9. Required the stock with which each commenced.

31. If a certain floor were 5 feet longer and 4 feet wider, it would contain 550 square feet. But if it were 4 feet longer and 5 feet wider, it would contain 192 square feet more than it actually does contain. Required the dimensions of the floor.

32. If A work 3 days and B 4, they will earn \$9; if A work 4 days and C 5, they will earn \$14; if B work 6 days and C 7, they will earn \$23. Required the daily wages of each.

33. Two numbers are in the ratio of 4 to 5, and the difference of their second powers is 81. What are these numbers?

34. The sum of two numbers is 18, and the sum of their squares is 164. Required the numbers.

35. What two numbers are those whose difference is 7, and

half of whose product increased by 30, is equal to the square of the less?

36. The product of two numbers is 120. Moreover, if 2 be added to the less, and 3 be subtracted from the greater, the product of the sum and difference will also be 120. Required the numbers.

37. A certain number of sheep cost £120; if 8 sheep more had been bought for the same sum, each would have cost 10s less. Required the number of sheep.

38. A, B and C had together £60; B, C and D had £90; C, D and A had £80; and D, A and B had £70. How much money had each?

39. A and B set out from the same place, and at the same time, to travel to a town at the distance of 300 miles. A goes 1 mile an hour more than B, and accomplishes his journey 10 hours sooner than B. At what rate does each travel?

40. A number consists of two digits. The left hand digit is three times the right; and if 12 be subtracted from the number, the remainder will be equal to the square of the left hand digit. Required the number.

41. A starts three hours and 20 minutes sooner than B, and travels uniformly 6 miles an hour. B starting from the same place follows at the rate of 5 miles the first hour, 6 miles the 2d, 7 miles the 3d, and so on. In what time will B overtake A?

42. Two men, 93 miles apart, set out at the same time to meet. One commences at 3 miles an hour, and increases his rate 2 miles each hour; the other commences at 15 miles an hour, and diminishes his rate 3 miles each hour. In how many hours will they meet?

43. There are two numbers, such that 27 times the greater is equal to the square of 27 times the less; and 3 times the greater is equal to the cube of 3 times the less. What are these numbers?

44. A and B each bought a farm; A's farm exceeded B's by 4 acres; each gave as many cents per acre as there were acres

in the farm which he bought; and both together paid \$816·16. How many acres did each buy?

45. Find two numbers, such that the square of the greater multiplied by the less shall be equal to 100, and the square of the less multiplied by the greater shall be equal to 80.

46. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 is to the greater. Required the numbers.

47. A rectangular house lot, whose length exceeds its breadth by 50 feet, contains 15000 square feet. Required the dimensions.

48. The sum of the second powers of two numbers is 244, and the second power of their sum is 484. What are the numbers?

49. The breadth of a rectangular field is to its length as 4 to 5. It is worth twice as many cents per square rod as there are rods in breadth, and the worth of the whole is \$1600. Required the dimensions.

50. The sum of two numbers added to a mean proportional between them is 37; and the sum of the squares of the numbers added to their product is 481. Required the numbers.

51. The sum of two numbers multiplied by their product is 240; and their difference multiplied by their product is 48. Required the numbers.

52. Separate 24 into two such parts, that the product of these parts shall be to the sum of their second powers as 3 to 10.

53. The sum of two numbers multiplied by the square of their product is 1800; and the difference of the numbers multiplied by the square of their product is 450. Required the numbers.

54. Find two numbers, such that the difference of their squares shall be 56; and $\frac{1}{3}$ of their product added to the square of the less shall make 40.

55. In a certain school, the number studying geometry is the square root of the whole number of scholars; $\frac{4}{9}$ of the whole learn algebra; and 36 scholars learn arithmetic. These three

classes constitute the whole school. Required the whole number of scholars.

56. A number, consisting of two digits, being multiplied by the left hand digit, produces 46; but if the sum of the digits be multiplied by the same digit, the product will be 10. What is the number?

57. There are two rectangular vats, whose cubical contents differ by 20 feet. The bottom of each is a square, one side of which is equal to the depth of the other vat; and the capacities of the two vats are as 4 to 5. Required the depth of each.

58. What number is that, from which if 4 be subtracted, this remainder shall exceed its square root by 2?

59. What number is that, to which if 24 be added, and the square root of this sum be extracted, this root shall be less than the original number by 18?

60. A board fence was built round a rectangular court to a certain height. The length of the court was 8 times the height of the fence wanting 2 yards; its breadth, 6 times the height of the fence wanting 5 yards; and the area of the court exceeded that of the fence by 178 square yards. Required the height of the fence and the dimensions of the court.

61. A man bought a quantity of cloth for $60. If he had bought 3 yards more for the same money, it would have cost $1 a yard less. How many yards did he buy?

62. Two men, A and B, set out at the same time, the former from the town C, and the latter from the town D, and travel towards each other. When they met, A had gone 30 miles more than B; and according to the rate they had traveled, A could reach D in 4 days, and B could reach C in 9 days, from the time of meeting. Required the distance between the towns.

63. What number exceeds its square root by 20?

64. Two retailers, A and B, jointly invested $500 in business. A's money was employed 5 months, B's only 2 months, and each received $450 for his capital and gain. How much money did each advance?

65. Find two numbers, whose difference added to the difference of their squares, makes 150, and whose sum added to the sum of their squares, makes 330.

66. Find a number consisting of three digits, such that the sum of the squares of the digits shall be 66; that the square of the middle digit shall exceed the product of the other two by 9; and, if 594 be subtracted from the number itself, the digits shall be inverted.

67. Five gamesters, A, B, C, D, and E, play together, on condition that he who loses, shall forfeit to all the rest as much money as they already have. First A loses, then B, then C, then D, and finally E. Yet, at the end of the fifth game, each has left $32. How much has each at first?

68. A and B sold 100 eggs, and each received the same sum. If A had sold as many as B, he would have received 18 pence for them; and if B had sold as many as A, he would have received only 8 pence for them. How many did each sell?

69. Separate 24 into two parts, whose product shall be 35 times their difference.

70. What two numbers are those, whose product is 4 times their difference, and whose product multiplied by their difference is 16?

71. The sum of three numbers is 21; if the first be subtracted from the second, and the second from the third, the latter remainder will exceed the former by 3; moreover, the sum of the squares of the first and third is 137. Required the numbers.

72. There are two rectangular vessels, which together hold 180 cubic feet; the bottom of each is a square whose side is equal to the height of the other vessel. If each vessel were a cube whose side was equal to one side of its bottom, the two vessels would contain 189 cubic feet. Required the dimensions of each.

73. There are two numbers, such that the square of the greater, multiplied by the less, is 30 more than the square of the

less, multiplied by the greater; moreover, the 3d power of the greater exceeds that of the less by 98. What are the numbers?

74. What number is that whose fourth power exceeds ten times its second power by 936?

75. Find a number, such that if its square root be increased by 4, the cube root of the sum shall be 2.

76. The first year a man was in trade he doubled his money; the second year he gained $5 more than the square root of the number of dollars he had at the commencement of that year, when he received a legacy of as many dollars as were equal to the square of the number he then had, and found that his whole fortune amounted to $13340. With how much money did he commence business?

77. If the sum of two numbers be increased by 2, and the second power of this result be added to the sum of the numbers, the amount will be 154. Moreover, the difference between the second powers of the two numbers is 40. What are the numbers?

78. The sum of three numbers in progression by difference is 15; and the sum of the squares of the extremes is 58. Required the numbers.

79. Four numbers are in progression by difference; the sum of the squares of the first two terms is 10; and the sum of the squares of the last two terms is 74. What are the numbers?

80. Find three numbers in progression by quotient, whose sum is 26, and the sum of whose second powers is 364.

81. Four numbers are in progression by quotient; the sum of the first two is 30, that of the last two is 120. Required the numbers.

82. Required the compound interest on £120, for 10 years, at 6 per cent. annually.

83. What will $300 amount to in 10 years, at compound interest semi-annually, the yearly rate being 5 per cent.?

THE END.